slight modification of this proof and shows how the modified form of reasoning may be applied to prove that the equilateral triangle circumscribing a given circle has a shorter perimeter and a smaller area than any other triangle circumscribing the same circle. These proofs make no use of parallels and hence apply without the Euclidean restriction that the sum of the angles of a triangle shall be 180.°

PAUL R. RIDER, *Secretary.*

THE FOURTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The fourth annual meeting of the Philadelphia Section of the Mathematical Association of America was held in Bennett Hall at the University of Pennsylvania on Saturday, November 30, 1929. There were two sessions, Professor A. H. Wilson of Haverford College presiding at both sessions.

The attendance was fifty-one, including the following thirty-five members of the Association: V. W. Adkisson, E. F. Allen, P. A. Caris, G. G. Chambers, J. W. Clawson, E. S. Crawley, J. E. Davis, Fletcher Durrell, L. P. Eisenhart, Michael Goldberg, O. E. Glenn, J. C. D. Harding, G. A. Harter, Frances H. Jackson, R. W. Jones, C. A. Keeler, J. R. Kline, M. S. Knebelman, P. A. Knedler, K. W. Lamson, W. F. Long, H. M. Lufkin, L. D. McDonough, A. E. Meder Jr., H. H. Mitchell, Richard Morris, C. A. Nelson, A. G. Rau, C. J. Rees, J. F. Ritt, J. H. Roberts, J. A. Roulton, George Rosengarten, Pincus Schub, A. H. Wilson.

At the business meeting the following officers were chosen for next year: Chairman, Professor J. A. MILLER, Swarthmore College; Secretary, P. A. CARIS, University of Pennsylvania; Program Committee, Professors Miller (ex-officio), Fort and Kline. The next meeting of the Section will be held on Saturday, November 29, 1930 at the University of Pennsylvania.

The following papers were presented:

1. "Wave mechanics," by Professor K. W. LAMSON, Lehigh University.
2. "Group characters," by Professor H. H. MITCHELL, University of Pennsylvania.
3. "Dynamical trajectories and geodesics," by Professor L. P. EISENHART, Princeton University.
4. "Integration in finite terms," by Professor J. F. RITT, Columbia University.

Abstracts of the papers follow below, the numbers corresponding to the numbers of the titles:

1. The paper by Professor Lamson mentioned some types of experiments which led to the use of the quantum theory. In that theory only integral values are allowed for some of the variables, which in ordinary mechanics could take on any of a continuous set of values. Schrödinger shows how to replace this assumption by a less startling one. He develops a mechanics whose difference

from ordinary mechanics is analogous to the difference between wave optics and geometrical optics. Corresponding to energy in mechanics is frequency in optics. The whole numbers must enter the theory at some point and Schrödinger gets them by the mild postulate that certain functions must be finite for all values of the variables.

2. The theory of group characters, as developed by Frobenius, Schur, Burnside, and others, plays a fundamental role in the problem of the representation of a given abstract group as a group of linear substitutions. Each such representation that is irreducible determines a set of sums of roots of unity, the "traces" of the substitutions that correspond to the different conjugate sets of operators of the given group. A set of numbers of this sort is termed a "character" of the group, and the number of distinct characters is equal to the number of conjugate sets of operators. A variety of interesting relations hold between the numbers that constitute these characters, by use of which much light can be thrown on the representations of the given group as a group of linear substitutions. An application of the theory to abstract groups is the proof, due to Burnside, that no group whose order is divisible by just two different primes can be simple. Expositions of the theory are now available in several texts.

3. Professor Eisenhart's paper has been published in full in the October 1929 number of the *Annals of Mathematics*.

4. Professor Ritt discussed Liouville's work on the impossibility of performing certain integrations, and solving certain differential equations, in finite terms.

P. A. CARIS, *Secretary*.

THE UNDERGRADUATE MATHEMATICAL CURRICULUM OF THE LIBERAL ARTS COLLEGE*

By F. L. GRIFFIN, Reed College

1.

No curriculum can be intelligently planned, in any subject, without careful consideration of the *human needs*, individual and social, which should be met by instruction in and study of that subject. Colleges of engineering, commerce, and other specialized professions face rather specific demands which largely prescribe their curricula. Even they, however, may well be influenced by the more general considerations which should govern in the college of liberal arts and science. Consequently, portions of the following discussion apply in a measure even to the specialized schools.

The general educated public has as yet scarcely any conception of the rôle which mathematics has played in the development of our present civilization, is playing now in the rapid advance of many sciences, including the life sciences and some social sciences, and is destined to play further in the creation of a more

* Read in substantially this form at the summer meeting of the Mathematical Association of America at Boulder, Colorado, Aug. 26, 1929.

hopeful and rational social order, the enrichment of esthetics and art, and the full liberation of the human spirit.

In all the physical, engineering, and earth sciences, mathematics is an indispensable tool. We are told by one competent to speak that "chemistry has graduated from the class of descriptive sciences into the class of exact sciences and has taken its place by the side of physics and engineering as a branch of mathematics. All chemical phenomena and many biological phenomena are now being interpreted in terms of physical chemistry, and physical chemistry cannot be mastered without a foundation in mathematics which extends through calculus."[1] In biology differential equations are used in studying growth and senescence, digestion, healing and other metabolic activities; elementary calculus, graphical and statistical methods in studying biometry, genetic equilibrium, surface tensions, energetics, and so on. In some biological journals calculus is becoming conspicuous.[2] Similar mathematical methods are rapidly coming into use in psychology and education[3] for the study of foveal vision, the psychology of musical ability, mental growth and learning curves, threshold stimuli for sensations, the Weber-Fechner law, the distribution of mental ability, special types of tests,[4] and various other topics in the new science of psychophysics. A recent writer's allusion to differential equations as a suitable means of expressing the *interaction* of individual and environment in determining human conduct and social evolution, shows the current trend of thought and the possibility of a clarification similar to that introduced into economics by Cournot, Jevons, Walras, and others more recently, through mathematical thinking and the formal use of calculus. The theory of maxima and minima has brought to the business world a great economizing of materials. The theory of probability also is effecting important economies, besides rendering invaluable service in many fields.[5] Some knowledge of the mathematics of finance, and of calculus, is becoming indispensable in handling intricate questions of investment, *e.g.*, long leases with varying rentals and adjustments in the event of building; serial bonds with split rates of interest; etc. Sociological thinking, too, is not only being based increasingly upon statistical methods, but is beginning to be affected fundamentally by the conceptions of the calculus.[6]

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[1] F. Daniels, *Mathematics for students of chemistry*, this Monthly, vol. 35 (1928), pp. 3–9.

[2] O. W. Richards, *The mathematics of biology*, this Monthly vol. 32 (1925), pp. 30–36. See also T. B. Robertson, *The chemical basis of growth and senescence* (Lippincott, 1923); also his *Principles of biochemistry*; and recent volumes of The Journal of General Biology, The Journal of Comparative Neurology, and The Journal of Experimental Medicine.

[3] See recent volumes of The Journal of Experimental Psychology and The Journal of Educational Psychology.

[4] H. M. Walker, *Certain mathematical questions suggested by the true-false test*, this Monthly, vol. 34 (1927), pp. 503–15.

[5] T. C. Fry, *Probability and its engineering applications*. (Van Nostrand, 1928).

[6] Bureau of Census, U. S. Life Tables (1921), pp. 329 ff. See also F. S. Chapin, *Cultural Change* (The Century Co., 1929), pp. 357–384; also R. Pearl, *Biology of Population Growth*, and bibliography (Knopf, 1925).

Among the arts we note the architectural and decorative uses of geometry, the mathematical basis for the choice of graceful curves in drawing and painting,[7] the physico-mathematical basis for the communication of emotion through music, for the criticism of instruments and technique, and for the introduction of new scales.[8] The improved electric recording of music is a notable advance based partly upon the study of differential equations. In games, puzzles and cryptography, mathematics is frequently basic. And now comes a "calculus of variants" to the assistance of the student of literary criticism![9] May we remark further that mathematical manipulation is itself a fine art, requiring skill, judgment, and a sense of appropriateness comparable with like qualities requisite for success in the other arts?

All of this merely hints at what mathematics has contributed thus far toward our external or objective progress. It is also basic for the inner life of speculative philosophy: both as to the problem of knowledge and as to the nature of the universe. Touching the former, it provides a norm of logical structure and rigor, shows the unreliability of intuition in even apparently simple matters, and discloses the relative character of all knowledge. Touching the latter, it exhibits possible varieties of space, generalizes number concepts, clarifies the infinite, the nature of law or functionality, and determines invariants. It even touches questions of religious attitudes and spiritual values.[10]

This progress has been made despite the fact that only a small part of the workers in many fields have had any considerable familiarity with the concepts and methods of mathematical analysis. Doubtless the infiltration of mathematics into some fields will be slower than it has been into fields now saturated with our science. But who can confidently delimit the totality of subjects susceptible of helpful mathematical treatment? May we not reasonably anticipate a great surge of explanation and discovery in many fields when the knowledge and spirit of mathematics shall have seized upon most of the workers?

2.

To most students with other demands upon their time and interest, the freshman course presents the last opportunity for formal mathematical study. Whatever help we are to give the great mass, who will include many educational leaders of the next few decades, must be given in the freshman course. One reason for the past and present general ignorance of the possibilities of mathematics on the part of educated laymen is that there has been *scant opportunity* to acquire the requisite familiarity in the time available. Introductory courses have not adequately exhibited the possibilities of our science, nor sufficiently covered its major concepts and methods, and the latter have been obscured by

[7] J. Ruskin, *The elements of drawing and the elements of perspective*, pp. 155 ff. (Dutton, 1907). See also this Monthly, vol. 25 (1918), p. 192.

[8] J. M. Barbour, *Synthetic musical scales*, this Monthly, vol. 36 (1929), pp. 155–160. See also references given by R. C. Archibald in this Monthly, vol. 31 (1924), pp. 1–25.

[9] W. W. Greg, *The calculus of variants, an essay on textual criticism*. (Clarendon Press, 1927).

[10] D. E. Smith, *Religio mathematici*, this Monthly, vol. 28 (1921), pp. 339–349.

association with topics and technique unnecessary to the proposed objective and better postponed to later courses. Professor Whitehead's warning against reconditeness is much to the point; as is also his statement that "education to be living and effective must be directed to informing pupils with those ideas, and to creating for them those capacities which will enable them to appreciate the current thought of their epoch."[11]

The freshman course can render the great and important service needed only if it includes a thorough treatment of graphs as an instrument of calculation, the elements of differential and integral calculus with a considerable variety of applications, at least enough trigonometry to solve triangles and understand simple periodic oscillations, enough analytic geometry to see its power, solve simple locus problems, and encounter the several conics and some other curves, enough work in logarithmic calculation and in the approximate solution of higher numerical equations to handle these matters effectively, at least the basic ideas of statistics and probability, and some work on investments. Various other topics are desirable, but less essential. The graphical work should stress calculations made by drawing tangents and by measuring areas: in chemistry these have "brought order out of chaos in the study of solutions,"[1] and in many other fields where functional formulas are unknown, they are the sole reliance. The work in calculus should include partial differentiation, which is basic for the understanding of many subjects, from chemical thermodynamics to least squares in educational measurements. Integration should be applied in so many fields that its wonderful possibilities and generality will be sensed, and the procedure for applying it be grasped. "The ability to think in mathematical terms is absolutely essential." Even the work in differentiation falls far short of fruition if integration be slighted.

Such a freshman course can give in a single year a comprehension and working knowledge of elementary mathematical analysis sufficient for studies in many fields. But to deepen the student's insight and extend his horizon, the student's work in the course should be *supplemented by brief lectures* on the significance of important ideas and processes, and of mathematical analysis as a mode of thought, historical matters, the nature of mathematics as a logical system, its universality and some of its philosophical implications. the logical side of algebra, the parallel postulate and possibly some references to relativity. By all means, a simple explanation should be given of the *idea* of a differential equation of the first order: how it expresses the determination of the rate of change of one varying quantity by some other variable and perhaps by the first quantity itself; what is meant by solving the equation; and so on. (We may fairly question whether a fully satisfactory textbook for such a freshman course has yet been published; but, with continued experimentation and revised editions by various authors, much improvement may doubtless be expected.)

The sophomore course, while serving a much smaller clientele, should again be planned as generally as is feasible. Beyond this the courses will naturally be

[11] A. N. Whitehead, *The aims of education and other essays*, p. 116 ff. (Macmillan, 1929).

more specialized to meet the needs of particular groups. The broad, unified freshman course will prove to be a thoroughly adequate foundation for further study only if it is followed by a course definitely based upon it, which utilizes its gains and fills any gaps left in the freshman year. The greatest need will be drill in technique and practice in making still further applications. To learn to employ calculus in new situations a student must see it at work, and must put it to work, in a variety of fields. Whatever we may think of the doctrine of no-transfer-of-training, we can be sure that the difficulty of applying a familiar process in a field of new ideas will be greatly lessened if it has in the past been applied to novel situations involving similar elements. Thus the calculus should be seen at work not only in geometry and mechanics but also in other branches of physics, in chemistry, in meteorology, in physiology and psychology, in economics and actuarial science. There should be extensive practice in setting up integrals other than those already analyzed in the text, which the student merely reproduces with new limits and new functional formulas to fit the particular case. That is, the student should have actual practice in *analyzing ideas* and reasoning out the type form of the integrand. (He should learn to use the rapid, vigorous, abridged statement with "tiny elements," but should be able to justify his conclusions by a sound limit procedure.) Likewise he should have considerable practice in formulating differential equations as well as in solving them. In applying integration to geometry, he should handle many combinations of elementary surfaces for which no equations are given,—either getting the equations for himself or actually *seeing* the limits of integration from drawings of suitable sections. He should be able to test directly the consistency of two supposed integrals of a common integrand; should learn when to use tables of integrals and when not,—the most efficient procedure always,—and should become familiar with ordinary tables of probability, elliptic integrals, and the gamma function. An introduction to hyperbolic functions is highly important, as are also a brief introduction to Fourier series, and a discussion of mean values for different independent variables, or over a non-linear region. Partial derivatives deserve careful treatment; likewise curvature and motion. The purely geometrical properties of conics may well be postponed for study in the course next to be discussed. (Before leaving the present topic, however, may we ask whether the two-year course so far outlined would not be as satisfactory for engineering students as in a college of liberal arts and science? *The research papers by numerous seniors*, mentioned later, *apparently indicate that unified courses give an adequate foundation for specialized work.*)

3.

In the junior and senior years, at least the following specialized courses are needed to serve various particular groups of students. (Some of these will serve more than one group.) After each title is suggested a suitable number of "units" i.e., year-hours, or the equivalent.

(*a*) *Modern Geometry, Analytic and Synthetic* (*4*). Elementary courses in analytic geometry are generally more "analytic" than they are "geometry"; by

the time the student has learned the basic formulas and equations, and has sensed the method, there is little time left to use the new instrument for the actual investigation of geometric questions. The junior course may well start with a brief concise summary of analytic method, and then apply the methods to *a genuine study of geometry*. The conics offer a veritable mine of theorems relating to diameters, tangents, polars, etc. Several important higher plane curves should be studied, starting from their locus definitions, determining their shapes and special properties, and then returning to them presently in a systematic treatment of the inversion of conics, and pedals of various curves. Much use should be made of construction, accurate or rough, so that the student may see the loci develop, point by point. Likewise, in later parts of the course. This analytic work which vastly enlarges the student's conception of geometry, need not take more than a third of a year. It may well be followed by a brief introduction to projective geometry, chiefly synthetic, occupying even less time, but opening to the student a whole range of new ideas, including Pascal's, Brianchon's and Desargues' theorems. The remaining portion of the year, from a third to a half, can hardly be spent more profitably than upon the type of modern geometry now commonly called "college geometry." This bears far more directly upon the problem of high school teaching than do most of the college courses.

(*b*) *Descriptive Geometry* (*4 or 2*) is another subject of great value to general students as well as engineers, because of its powerful method of studying spatial figures, the training in visualization, and its satisfying logical character. Prospective teachers of elementary or higher geometry should if possible study this subject, at least for a semester.

(*c*) *Higher Algebra* (*2*). Though the student may have had a brief introduction to determinants in his earlier work, and have applied them in his Modern Geometry, he can now advantageously study the general theory, including elimination. This together with work in the theory of equations, including simple group concepts and applications to the problem of the regular n-gon, trisection of the angle, etc., is important equipment for prospective high school and college teachers, and some others. One of my former students, a structural engineer, has found the general solution of the cubic to be of great value in his practice.

(*d*) *Advanced Calculus* (*4*) deserves the attention of mathematical and technical specialists and some others, throughout the senior year. More work in differential equations, ordinary and partial, with applications to vibrating strings and membranes, flow of heat, etc., may well receive a large portion of the time. This involves a definition and very brief study of the Bessel functions, and some further work in Fourier series. Time should also be given to the reduction of elliptic integrals, with many numerical examples and applications; and to devices for evaluating definite integrals, including differentiation with respect to a parameter. (It may be sufficient to state without proof a number of criteria as to the legitimacy of this and other processes, giving references to the litera-

ture where proofs can be found.)[12] There should be a brief introduction to the calculus of variations and to vector analysis, so that students for whom this is the last course in analysis may glimpse these fields, and those who go on may start the graduate courses more readily. Some attention should be given the geometry of surfaces and space curves; also to the complex variable; and to Green's and Stokes' theorems.

(*e*) *Probability and Mathematical Statistics* (*4 or 2*). In the junior and senior year some students will need a course in the calculus of observations. Besides the theory of probability and least squares, finite differences, empirical equations, and further statistical theory, it may cover special methods of numerical calculations.

(*f*) *Mathematics of Finance* (*2*) including insurance, and using the technical procedures not employed in the freshman course.

(*g*) *Spherical Trigonometry and Surveying* (*2*) with introductory work in solid geometry.

(*h*) *Astronomy* (*4*) unless offered in a separate department, should be available, at least a descriptive course with some mathematical calculations.

(*i*) *Analytic Mechanics* (*4*) is an excellent course; but in my opinion should be given in the physics department, illuminated by some experimental work.

(*j*) *History and Foundations* (*2*). At least a brief course of this sort should be a prerequisite for teaching elementary algebra and geometry. Many teachers of algebra are woefully ignorant of its axioms. (For example, recently, in some excellent high schools a number of experienced teachers were unable to supply inquiring pupils with a logical proof that $(-4)\ (-3)=12$. Instead of reasoning from axioms about *numbers*, they could only mumble about some alleged *analogies*!)

By the way, the "units" suggested for these courses need not necessarily be "hours"; but rather units of total time alloted to the course in class and out. At Reed College an elementary three-unit course may meet four times a week with short assignments, to allow relatively large supervision of the student's progress. An advanced four-unit course may meet but once a week, the student working independently for ten or more hours between. The number of meetings can also be varied during the course, or meetings intermitted, at the option of the instructor. The junior and senior work is under the jurisdiction of a "Division" of the faculty. All this makes for flexibility and effectiveness.

4.

The large college or university can offer a great variety of courses; the small institution has difficulty in providing for as much flexibility. A great help in this direction is to utilize some of the intellectual reserves of the student himself, particularly the advanced student, by offering some of the above courses (*e*)-(*g*) as Special Topics, for individual reading, practice and report. (Group work in

[12] E. J. Moulton, *The content of a second course in calculus*, this Monthly, vol. 25 (1918), pp. 429–34.

special topics for freshmen and sophomores may also profitably be given to supplement the regular courses, covering for instance special statistical methods, use of the slide rule, elements of determinants, some properties of conics, curve tracing with the help of calculus and such devices as Newton's triangle, imaginary regions, etc.) The more advanced work in special topics is substantially of the sort often called honors work, and enjoys the advantage of an engrossing interest, particularly when the student selects the topics himself from time to time.

In my judgment the most valuable experience a senior specialist student can have is to write a thesis, embodying such research as his capacity permits, on a suitably chosen individual problem, and take a final oral examination in his major field. In many institutions this type of work may have to be limited to a group of specially selected honors students. At Reed College, all seniors undertake this work, but to get senior standing a student must pass an intensive Qualifying Examination at the end of the junior year. At that, there is of course a wide variation in the ability of our seniors, and in the independence and value of their theses. An idea of the sort of topics which we have used as thesis subjects in mathematics may be obtained from the appended list.

In some institutions mathematical clubs may be of more value in maintaining student interest and developing initiative than the special methods I have mentioned. But there appears to be no mutual exclusiveness; both activities may be carried on if desired.

In closing let me again emphasize that one of our major tasks is to educate a great multitude in the concepts, modes of procedure, and criteria of mathematical analysis. This task recognized, and suitable action taken in organizing our freshman courses, mathematics will become the key subject in the curricula of the future. We are but at the dawn of the general utilization of mathematics in the amelioration of human life.

A List of Thesis Subjects

Nearly all of these bachelor theses, written at Reed College, embodied some new results of investigation; several broke ground virtually new. Those marked by an asterisk were read, in outline, before the San Francisco Section of the American Mathematical Society.

Integrals and Functions:

On the accuracy of certain approximations to the elliptic integrals.
On the reduction of quotients of quadratic radicals to elliptic integrals.
Imaginary substitutions for elliptic integrals of the first type.
A study of certain functions of a complex variable.
* Generalized functions of sectorial areas.
* Further generalization of the circular and hyperbolic functions of sectorial areas.

Calculus of Variations:

* Concerning geodesics on certain surfaces of revolution.
Some geodesic lines on an ellipsoid of revolution.
Geodesics on a right circular cone.
Some theorems in maxima and minima.

Geometry:

* Curves functionally conjugate with respect to a given curve and origin: Type I. (Others: Types II, III, IV, V, *VI.)

Some properties of normals to an ellipse.

* Certain families of curves having a parabola as their envelope. (Another: the cubical parabola.)

Certain properties of poles and polars.

* Conics through given points.

Concerning certain basic problems in analytic geometry. (Highly original.)

Investigations in the trisection of the angle.

Algebra

* An insolvable case of the generalized school-girl problem.

* Concerning the solution of cubic and quartic equations.

Equations of higher degree solvable by special artifices and devices.

The construction of a regular 257-gon.

Other Fields:

A mathematical theory of business profits (under special conditions).

* A study of caustic surfaces and of a certain image formed by reflection and refraction.

Accurate measurements with a steel tape or wire.

A new graphical method for determining the positions of a moving body under Newtonian attraction.

The advance of the apsidal lines in central orbits described in the equatorial planes of attracting ellipsoids of revolution.

Relation of secondary mathematics to higher mathematics.

Tentative plan for a course in first year mathematics for secondary schools.

* Some studies in probability and correlation.

THE PROBLEM OF TWO BODIES AND A RELATED SEXTIC[1]

By W. H. GARRETT, Baker University

1. *Equation of orbit.* Given two bodies with masses m_1 and m_2 which attract each other with a force which is proportional to the product of the masses and inversely proportional to the square of the distance between them, the differential equations of motion of one with respect to the other are:

$$(1) \qquad d^2x/dt^2 = -k^2Mx/r^3\ ;\ d^2y/dt^2 = -k^2My/r^3,$$

where $M = m_1 + m_2$, r = the distance between bodies, and k = acceleration at unit distance.

The integration of this system[2] gives as the polar equation of the orbit,

$$(2) \qquad r = c_1^2 k^{-2} M^{-1} / [1 - (1 + c_1^2 c_3 k^{-4} M^{-2})^{1/2} \cos(\theta - c_4)],$$

which is the equation of a conic with the origin at one of the foci.

The ordinary equation of a conic with the pole at the right hand focus is

$$(3) \qquad r = p/[1 + e \cos(\theta - \omega)],$$

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[1] Read before the Kansas Section of the Mathematical Association of America, February 4, 1928.

[2] For the separate steps in this integration, see F. R. Moulton's *Celestial Mechanics*, pp. 147–148.

where ω is the angle between the polar axis and the major axis of the conic.

Comparing (2) with (3) it is seen that

$$p = c_1^2 k^{-2} M^{-1}\ ;\ e^2 = 1 + c_1^2 c_3 k^{-4} M^{-2}\ ;\ \omega = c_4 - \pi. \tag{4}$$

2. *An important property of the orbit.* If $e<1$, the orbit is an ellipse and $p=a(1-e^2)$. Hence, combining this relation with the first two of equations (4), there is obtained:

$$c_3 = -\ k^2 M(1 - e^2)p^{-1} = -\ k^2 M a^{-1}.$$

Now one of the integrals of (1) is

$$(dx/dt)^2 + (dy/dt)^2 = 2k^2 M r^{-1} + c_3.$$

Hence the velocity V of one body relative to the other is given by the relation,

$$V^2 = k^2 M(2r^{-1} - a^{-1}).$$

Solving for a,

$$a = k^2 r M/(2k^2 M - V^2 r).$$

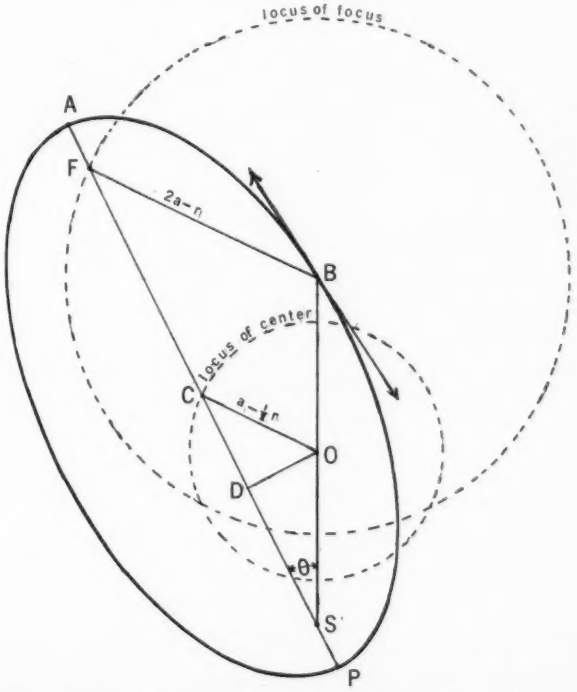

FIG. 1. S, common focus of all ellipses; B, point of projection of particle; F, second focus, A, aphelion point; P, perihelion point.

This establishes the following important property: If a body m_1 at a distance r from a body m_2 is projected in different directions with the same velocity V, $[V<(2k^2Mr^{-1})^{1/2}$, the parabolic velocity], then the orbits described by m_1 relative to m_2 will be ellipses with major axes equal in length.

3. *Loci of focus and center.* Let B (Fig. 1) be the point of projection of m_1, and S, the position of m_2, the fixed focus. The locus of the other focus F will be a circle with center at B and radius $2a-r$, since $SB+BF=2a$ and $SB=r$.

From this it follows that the locus of centers of all conics will be on a circle with center at O, the midpoint of SB, and radius $=a-\frac{1}{2}r$.

4. *Loci of Aphelion and Perihelion Points.*[3] In Fig. 1, let A be the aphelion point and P the perihelion point. Take the pole at S and let SB be the polar axis. Draw the perpendicular OD from O to the line SF. Then the loci of A and P in terms of the polar coordinates ρ and θ may be obtained as follows:

$$\rho = SD + DC \pm CA. \quad \text{Then, since } DC = (\overline{CO}^2 - \overline{DO}^2)^{1/2},$$

$$\rho = \tfrac{1}{2}r\cos\theta + [(a-\tfrac{1}{2}r)^2 - (\tfrac{1}{2}r\sin\theta)^2]^{1/2} \pm a,$$

which becomes

$$\rho^2 + 2a\rho + ar - r(\rho + a)\cos\theta = 0, \tag{5}$$

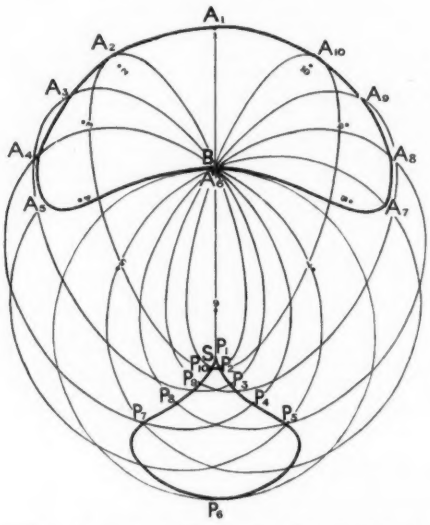

FIG. 2. $r=8$, $a=7$; S, common focus of all ellipses; B, point of projection; 1, 2, $\cdots$, 10, second foci; A_1, A_2, $\cdots$, A_{10}, aphelion points; P_1, P_2, $\cdots$, P_{10}, perihelion points.

[3] This problem appears in F. R. Moulton's *Celestial Mechanics*, page 155. It was however originally solved by the writer prior to the appearance of the first edition, while he was a member of Dr. Moulton's class in Celestial Mechanics.

which is the polar equation of the loci of the aphelion and perihelion points.

Changing to rectangular coordinates, the following sextic equation of the loci is obtained:

$$\begin{aligned} &x^6 - 2rx^5 + (r^2 + 3y^2 + 2ar - 4a^2)x^4 + (4a^2r - 2ar^2 - 4ry^2)x^3 \\ &\quad + (3y^4 + 4ary^2 - 8a^2y^2 + r^2y^2)x^2 + (4a^2ry^2 - 2ar^2y^2 - 2ry^4)x \\ &\quad + y^6 + (2ar - 4a^2)y^4 + a^2r^2y^2 = 0. \end{aligned} \tag{6}$$

In Figs. 2 and 3, the loci of the aphelion and perihelion points are constructed for the two cases $a<r$, and $a>r$. ($r=8$ and $a=7$ and 9).

Special case, $a=r$: If $a=r$ then equations (5) and (6) become

$$(\rho + r)(\rho + r - r\cos\theta) = 0,$$

$$(x^2 + y^2 - r^2)(x^4 - 2rx^3 + 2x^2y^2 - 2rxy^2 + y^4 - r^2y^2) = 0,$$

which represent the circle and the cardioid. These loci are shown in Fig. 4 and one may easily trace the transition from the loci of aphelion and perihelion

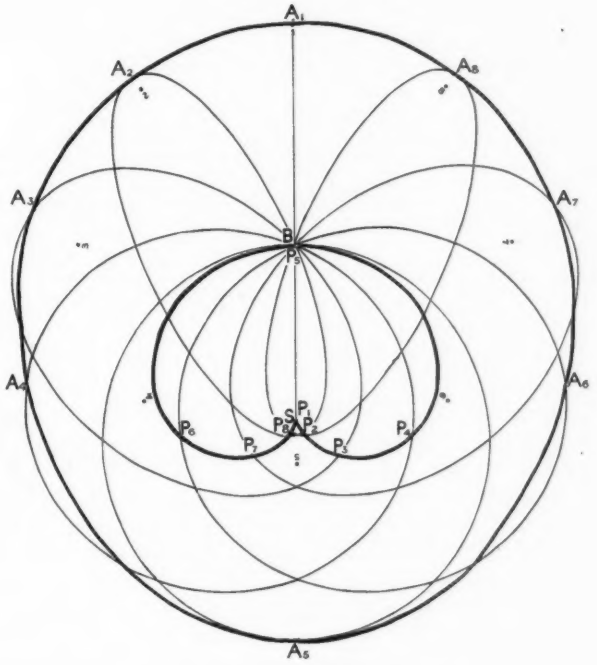

FIG. 3. $r=8$, $a=9$; other points as in FIG. 2.

points in Fig. 2 to the corresponding loci in Fig. 3 by means of this connecting link.

A further consideration of Fig. 4 also shows that here one member of the family of ellipses is a circle with radius r and any point on this circle may be considered as either a perihelion or aphelion point.

5. *Mechanical Construction of Sextic.* Several members of the family of sextics are shown in Fig. 5. These curves were constructed mechanically as follows:

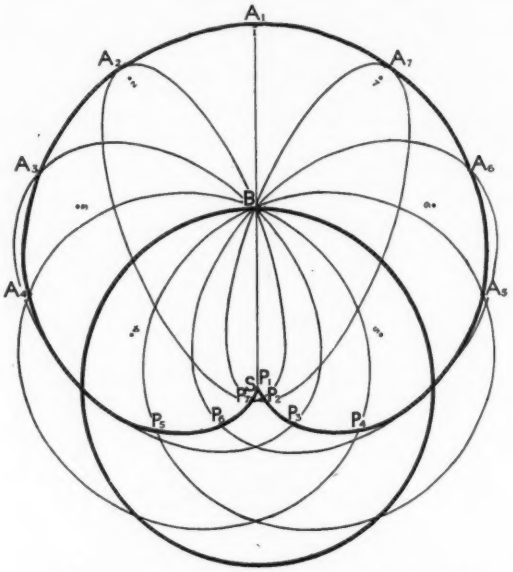

FIG. 4. $r=8$, $a=8$; other points as in FIG. 2.

Two pencils at A and P were held at a distance of $2a$ apart by means of two narrow strips of wood. Two stove bolts clamped the strips together, holding the pencils firmly in proper position.

A short cross arm pivoted at one end to a fixed nail at O[4] and at the other end to C, the midpoint of AP, allowed C to rotate around O at a distance of $a-\frac{1}{2}r$.

The groove between the parallel strips was placed over a fixed nail at S at a distance of $\frac{1}{2}r$ from O, and kept A, S, and P always on the same straight line, the major axis of the ellipse.

As the point C was revolved around O, the pencils at A and P described the locus of the sextic, that is the loci of the aphelion and perihelion points.

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[4] See Fig. 1.

The different members of the family were obtained by letting the parameter a take the values 6, $6\frac{1}{2}$, 7, $7\frac{1}{2}$, 8, $8\frac{1}{2}$, 9, with $r=8$.

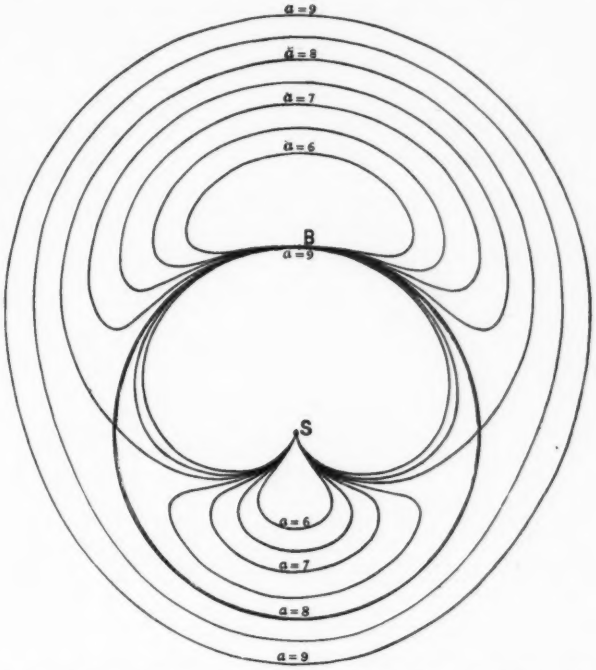

FIG. 5. Family of sextics for values $r=8$, $a=6$, $6\frac{1}{2}$, 7, $7\frac{1}{2}$, 8, $8\frac{1}{2}$, 9. In the special case $a=8$, the sextic becomes a circle and a cardioid.

THE RECTANGULAR HEXAGON

By PAUL WERNICKE, Washington, D. C.

The theorems here to be considered concern the figure of a hexagon in space (S_3), the consecutive sides of which meet at right angles. It is determined by three lines, P, Q, R, not in general all parallel to a plane. Following J. Petersen[1] we call P, Q, R the *sides* and the common perpendiculars (shortest distances) of the pairs QR, RP, PQ the *edges* of this "trilateral figure." The sides are determined by the edges as well as vice-versa, so that the hexagon may be considered as a trilateral figure in two ways.

The methods of proof are based on Study's coordinates of the straight line

[1] L'Intermédiaire des Mathématiciens, vol. 5 (1898) p. 36.

(Strahlen-Coordinaten[2]), the system of which, in turn, is built on Plücker's homogeneous (and redundant) coordinates of the right line as determined by two points, in Cartesian rectangular coordinates: $x=(x_1/x_0,\ x_2/x_0,\ x_3/x_0)$ and $x'=(x_1'/x_0',\ x_2'/x_0',\ x_3'/x_0')$:

$$rX_{0i}=x_0x_i'-x_ix_0',\quad rX_{jk}=x_jx_k'-x_kx_j',$$

where i, j, k are different, $i, j, k=1, 2, 3$, and r is a factor of proportionality.

Departing slightly from Study in order to use vector algebra,we combine the first three into the *vector* of X:

$$X_v=iX_{01}+jX_{02}+kX_{03};$$

and the last three into its *moment* (about the origin of coordinates):

$$X_m=iX_{23}+jX_{31}+kX_{12}.$$

We further write $X=X_v+\epsilon X_m$, where ϵ is a symbol to be defined.
Since the determinant

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_0' & x_1' & x_2' & x_3' \\ x_0 & x_1 & x_2 & x_3 \\ x_0' & x_1' & x_2' & x_3' \end{vmatrix}$$

vanishes we find that the scalar product $X_v\cdot X_m=0$. This is Plücker's equation. If

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_0' & x_1' & x_2' & x_3' \\ y_0 & y_1 & y_2 & y_3 \\ y_0' & y_1' & y_2' & y_3' \end{vmatrix} \equiv X_v\cdot Y_m+X_mY_v=0,$$

the lines X, Y intersect.

Dividing the rows of this determinant by $x_0,\ x_0',\ y_0,\ y_0'$, respectively, and subtracting the third row from the fourth, then the first from the second and third, we obtain the negative volume of the parallelopiped having x, x', y, y' for summits—negative, because $X_v,\ Y_v,\ X_v\times Y_v,\ Y_v$ form a positive sequence of vectors while the three rows referring to them in the determinant occur in the order $X_v,\ X_v\times Y_v,\ Y_v$. The base of this parallelopiped being $X_v\times Y_v$ its altitude is the distance of Y from X.

$$X_v\cdot Y_m+X_m\cdot Y_v=-\sqrt{X_v^2}\sqrt{Y_v^2}\sin(X_v,Y_v)\operatorname{dist}(X,Y).$$

Now if the components of the two vectors in $P=P_v+\epsilon P_m$ are *any* six quantities, not in general satisfying Plücker's equation, then, with an indeterminate

[2] Study, Géométrie der Dynamen, Leipzig (1903), p. 201.

line X (for which $X_v \cdot X_m = 0$), the equation $P_v \cdot X_m + P_m \cdot X_v = 0$ is well known to represent a *nullsystem* (a "screw" in R. S. Ball's terminology).

Putting $U_v = P_v^3$ and $U_m = P_v^2 P_m + (P_v \cdot P_m) P_v$ we have $U_v \cdot U_m = 0$. There exists, therefore, a line $U = U_v + \epsilon U_m$. It is the principal axis of the null system, and remains unchanged if P be replaced by $P' + aP_v + \epsilon(aP_m + bP_v)$, for $U_v' = a^3 P_v^3$ and $U_m' = a^3[P_v^2 P_m - (P_v P_m) P_v]$, and division by a^3 is permissible. We must have $a \neq 0$ but otherwise a, b may be any (real) numbers.

Defining ϵ as a unit in the nature of an infinitesimal, so that $\epsilon^2 = 0$ while the first power of ϵ must be carried in the work, P' arises from P through multiplication by $a + \epsilon b$, a complex number which we call a *scalar dualite* while $P = P_v + \epsilon P_m$ is a *vectorial dualite*.[3]

Multiplying P by any scalar dualite $a + \epsilon b$ (where $a \neq 0$), we obtain a null system coaxial to P. Study now takes the step of using P, subject to such multiplication, as the representative of the right line which is the axis of $P_v \cdot X_m + P_m \cdot X_v = 0$.

The dualite components of P,

$$P_1 = P_{01} + \epsilon P_{23}, \quad P_2 = P_{02} + \epsilon P_{31}, \quad P_3 = P_{03} + \epsilon P_{12}$$

are *Study's coordinates* of the right line (Strahlencoordinaten). Multiplication by

$$P_v^2 + \epsilon(P_v \cdot P_m)$$

at any time effects a reversion to Plücker's coordinates. The *scalar product of vectorial dualites* $P \cdot Q$ is

$$P \cdot Q = (P_v + \epsilon P_m)(Q_v + \epsilon Q_m) = P_v \cdot Q_v + \epsilon(P_v \cdot Q_m + P_m \cdot Q_v).$$

$P \cdot Q = 0$ carries with it the separate vanishing of the terms $P_v \cdot Q_v$ (meaning perpendicularity) and $P_v \cdot Q_m + P_m \cdot Q_v$ (meaning intersection). P and Q, therefore, intersect at right angles. *The vectorial product*,

$$P \times Q = P_v \times Q_v + \epsilon(P_v \times Q_m + P_m \times Q_v)$$

represents the common normal of P and Q since $P_v \cdot (P \times Q) = Q_v \cdot (P \cdot Q) = 0$, as is readily verified. We have $P \times Q = -Q \times P$, whence $Q \times Q = 0$.

For $P \cdot (Q \times R) = Q \cdot (R \times P) = R \cdot (P \times Q)$ we write more briefly (with Study), (PQR). The vanishing of this scalar product evidently means that P, Q, R have a common normal.

$$\begin{aligned} P \times (Q \times R) = {} & P_v \times (Q_v \times R_v) + \epsilon[P_v \times (Q_v \times R_m) + P_v \times (Q_m \times R_v) \\ & + P_m \times (Q_v \times R_v)] = Q_v(R_v \cdot P_v) - R_v(P_v \cdot Q_v) + \epsilon[Q_v(R_m \cdot P_v) \\ & - R_m(P_v \cdot Q_v) + Q_m(R_v \cdot P_v) - R_v(P_v \cdot Q_m) + Q_v(R_v \cdot P_m) \\ & - R_v(P_m \cdot Q_v)] = Q_v(R \cdot P) - R_v(P \cdot Q) + \epsilon[Q_m(R_v \cdot P_m) \\ & - R_m(P_v \cdot Q_v)], \end{aligned}$$

[3] Study speaks of "dual numbers" and uses the terms "scalar" and "vectorial" in a different meaning.

whence by the addition of $\epsilon^2[Q_m(R_v \cdot P_m + R_m \cdot P_v) - R_m(P_v \cdot Q_m + P_m Q_v) = 0$ we get $P \times (Q \times R) = Q(R \cdot P) - R(P \cdot Q)$, the *common normal* of P and $Q \times R$.

The sides of our rectangular hexagon or "trilateral" being P, Q, R, its edges become $Q \times R$, $R \times P$, $P \times Q$. The theorem that the three normals between side and opposite edge have a common normal was proved by F. Morley[4] in 1898 and announced by J. Petersen[5] in the same year, to be proved later.[6]

The vector product of two of these normals is

$$[Q \times (R \times P)] \times [R \times (P \times Q] = (Q \times R)(R \cdot P)(P \cdot Q) + (R \times P)(P \cdot Q)(Q \cdot R) + (P \times Q)(Q \cdot R)(R \cdot P),$$

an expression obtained by Study for the common normal of the three which he calls Morley's line. Its invariance under transpositions of P, Q, R shows that it is the normal of any two of the lines $P \times (Q \times R)$, $Q \times (R \times P)$, $R \times (P \times Q)$.

A theorem on a *type of perspective* of two rectangular hexagons is given by J. Petersen in the same volume of L'Intermédiaire:

When in two trilaterals the three normals of corresponding sides have a common normal, those of the corresponding edges have one likewise.

For a proof let the sides of the first hexagon be P, Q, R, its edges, $Q \times R$, $R \times P$, $P \times R$, while the sides of the other are denoted by $Q' \times R'$, $R' \times P'$, $P' \times Q'$ and its edges by P', Q', R'. Normals of corresponding sides are:

$$P \times (Q' \times R'), \quad Q \times (R' \times P'), \quad R \times (P' \times Q')$$

and of corresponding edges:

$$P' \times (Q \times R), \quad Q' \times (R \times P), \quad R' \times (P \times Q).$$

The scalar products:

$$(P \times [Q' \times R'], Q \times [R' \times P'], R \times [P' \times Q']) = (P'Q'R')[(Q' \cdot R)(R' \cdot P)(P' \cdot Q) - (Q \cdot R')(R \cdot P')(P \cdot Q')]$$

$$(P' \times [Q \times R], Q' \times [R \times P], R' \times [P \times Q]) = (PQR)[(Q \cdot R')(R \cdot P')(P \cdot Q') - (Q' \cdot R)(R' \cdot P)(P' \cdot Q)]$$

evidently vanish together. This proves the proposition.

The *cosine of the angle* between two *vectors* Q_v, R_v is given by the expression $Q_v \cdot R_v / (\sqrt{Q_v^2}\sqrt{R_v^2})$. We correspondingly form $Q \cdot R / (\sqrt{Q \cdot Q}\sqrt{R \cdot R})$ where the denominator may be written $\sqrt{Q_v^2}\sqrt{R_v^2}$.

$$Q \cdot R = \cos(Q_v, R_v) - \sin(Q_v, R_v) \operatorname{dist}(Q, R).$$

Accordingly, Study defines a *dualite angle*:

$$\operatorname{ang}(Q, R) = \operatorname{ang}(Q_v, R_v) + \epsilon \operatorname{dist}(Q, R)$$

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[4] Proceedings of the London Mathematical Society vol. 29 (1898), p. 670.

[5] L'Intermédiaire des Mathématiciens vol. 5 (1898), p. 36.

[6] Nyt Tijdskrift for Mathematiker vol. 2 (1899).

of which $Q \cdot R - \sqrt{Q \cdot Q}\sqrt{R \cdot R}$ is the cosine. Its sine can either be found directly or by the usual formula $\sin x = \sqrt{(1-\cos^2 x)}$. It is

$$\sqrt{[(Q \times R) \cdot (Q \times R)]} \div [\sqrt{(Q \cdot Q)}\sqrt{(R \cdot R)}].$$

A "transversal" line U divides the angle ang (Q, R) in the sine-ratio:

$$\begin{aligned} &\sin \text{ang } (Q, U)/\sin \text{ang } (U, R) \\ &= \sqrt{[(Q \times U) \cdot (Q \times U)]}\sqrt{(R \cdot R)}/\sqrt{[(U \times R)(U \times R)(Q \cdot Q)]} \\ &= -\sin \text{ang } (Q, U)/\sin \text{ang } (R, U). \end{aligned}$$

Evidently *the three sine-ratios in which the angles ang* (Q, R), *ang* (R, P), *ang* (P, Q) *are divided by* U *have negative unity for their product.* This corresponds to the theorem of Ceva for plane triangles and, when stated for the dualite angles between the edges instead of the sides, to the theorem of Menelaus.

We have, thus, a system of perspective based on that of plane geometry, in which *intersections* of lines are replaced by their *common normals* and the rectangular hexagon (trilateral figure) takes the place of the triangle. As Petersen suggests, a projective geometry may be based thereon.

A PROBLEM IN THE CALCULUS OF VARIATIONS

By C. E. RHODES, University of Cincinnati

In presenting the calculus of variations for the first time to a student, it is customary to give the well known classical examples of the minimum distance, the brachistochrone, and the minimum surface of revolution. Such problems can not be multiplied too much to give to the beginner a thorough understanding of certain ideas which are later generalized. It should therefore be of interest to present an additional problem which not only has a concrete setting, but which also brings out certain properties of the fields used in the sufficiency proofs.

A statement of the problem: Given a stream with straight parallel banks, such that the direction of flow is everywhere parallel to the banks. The velocity of the current is assumed to be an even function of the distance from midstream, having an absolute maximum, w_0, at midstream, and an absolute minimum, w_1, along either bank. A boat is to go from a fixed point on one bank to a fixed point on the opposite bank. If the speed of the boat relative to the water is a positive constant, u, determine the course of the boat so that the crossing can be made in the minimum time.

To set up the problem analytically, let us suppose the river to be 2 units wide. Let the distance between the projections of the end points of the course on the center line of the river be $2a$. (If the general direction of the course is downstream, a is positive; if upstream, a is negative. See Fig. 2.) If we take co-ordinate axes as shown in Fig. 2, the course will be symmetric with respect

to the origin, since the current, w, is an even function of x. We shall therefore consider only half the path, viz., from O to A. Let v denote the resultant of u and w, y the ordinate of any point on the course, s the distance along the course measured from O, θ the angle between v and the X-axis, and T the time required to go from O to A along the course. Projecting w and u onto v as

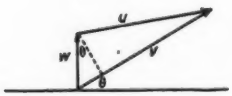

FIG. 1

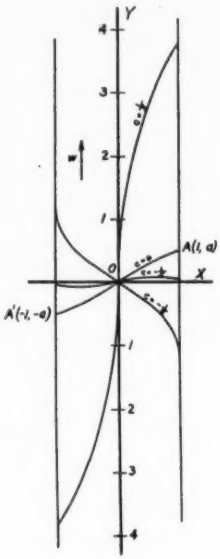

FIG. 2

shown in Fig. 1, we see that $v = w \cdot \sin\theta + \sqrt{(u^2 - w^2\cos^2\theta)}$. Since the resultant velocity, v, is the direction of the tangent to the course, $\tan\theta = dy/dx = y'$. Using this, we see that v may be put in the form

$$v = \frac{[wy' + \sqrt{(u^2 - w^2 + u^2y'^2)}]}{\sqrt{(1 + y'^2)}}.$$

From this it follows that

$$(1) \qquad T = \int_0^s \frac{ds}{v} = \int_0^1 \frac{(1 + y'^2)dx}{wy' + \sqrt{(u^2 - w^2 + u^2y'^2)}}$$

$$= \int_0^1 \frac{wy' - \sqrt{(u^2 - w^2 + u^2y'^2)}}{w^2 - u^2}\,dx.$$

Since by hypothesis u is a constant and w is a function of x alone, the integrand function in (1) is a function of x alone, and the Euler equation which determines our extremal curves becomes

$$w - [u^2y'/\sqrt{(u^2 - w^2 + u^2y'^2)}] = c(w^2 - u^2),$$

where c is a constant of integration. Solving this equation for y', and integrating again, we can express our two parameter family of extremals in the form

$$y = \int \frac{(1+cw)w + cu^2}{u\sqrt{[(1-cw)^2 - c^2u^2]}}\,dx + b\,,$$

b and c being the parameters. From our choice of axes, all members of this family must pass through the origin. This condition reduces the family of extremals to a one parameter family which can be written

$$y = \int_0^x \frac{(1-cw)w + cu^2}{u\sqrt{[(1-cw)^2 - c^2u^2]}}\,dx\,. \tag{2}$$

To insure a real integrand function, we must have

$$(1-cw)^2 - c^2u^2 = (1-cw+cu)(1-cw-cu) \geqq 0\,,$$

whence, if $w>u$, either

$$c \leqq 1/(w+u) < 1/(w-u) \qquad \text{or} \qquad c \geqq 1/(w-u) > 1/(w+u).$$

If $w<u$, as would usually be the case, either

$$1/(w-u) \leqq c \leqq 1/(w+u) \qquad \text{or} \qquad c \leqq 1/(w-u),\quad c \geqq 1/(w+u).$$

This last is impossible, and hence, considering the range of the variable, w, we have for $w<u$,

$$1/(w_1-u) \leqq c \leqq 1/(w_0+u). \tag{3}$$

For values of c within these limits, the integrand function is real, and equation (2) represents a real curve.

The second partial derivative with respect to y' of the integrand function in (1) turns out to be $u^2/(u^2-w^2+u^2y'^2)^{3/2}$ which is greater than zero for all values of x, y, y'. Hence both the Legendre and Weierstrass conditions are satisfied. For the Jacobi condition, there must be no point on the extremal between O and A that is conjugate to O, or, in other words, the particular extremal must not touch the envelope of the family (2) between O and A. From (2) we obtain

$$\frac{\partial y}{\partial c} = \int_0^x \frac{u\,dx}{[(1-cw)^2 - c^2u^2]^{3/2}}$$

Since this integrand function is always positive, $\partial y/\partial c$ never vanishes, save at $x=0$, there is no envelope, and hence no conjugate point. Furthermore, since $\partial y/\partial c$ is always positive, there can not be more than one member of the family (2) passing through any point, save at the origin. Hence there is a region of the plane simply covered by the family (2), and we have a direct proof of the existence of a field of extremals. Solving the equation,

$$a = \int_0^1 \frac{(1 - cw)w + cu^2}{u\sqrt{[(1 - cw)^2 - c^2u^2]}}dx,$$

for c, our solution must then be unique. Substituting this value of c in equation (2), we obtain the extremal curve which actually furnishes a strong relative minimum.

It is interesting to note that this field of extremals (2) may or may not cover the entire plane. The critical points occur when c is equal to one of its limits (3) for then the denominator in (2) vanishes. The field will have an upper bound if, and only if, the integral

$$\int_0^1 \frac{(1 - cw)w + cu^2}{u\sqrt{[(1 - cw)^2 - c^2u^2]}}dx$$

exists with $c = 1/(w_0+u)$, and a lower bound provided this same integral exists with $c = 1/(w_1-u)$. In the latter case the above integral takes the form

$$\int_0^1 \frac{(w_1 - w - u)w + u^2}{u\sqrt{(w - w_1 + 2u)}} \cdot \frac{dx}{\sqrt{(w - w_1)}}.$$

Hence a necessary and sufficient condition for the field to have a lower bound is that the integral $\int_0^1 dx/\sqrt{(w-w_1)}$ exist. In like manner, it follows that a necessary and sufficient condition for the field to have an upper bound is that the integral $\int_0^1 dx/\sqrt{(w_0-w)}$ exist. Experimental measurements[1] give for the current function of a straight stream the empirical form $w = w_1 + (w_0 - w_1)\sqrt{(1-x^2)}$. With this function, our field of extremals would have a lower bound, but no upper bound.

By making the substitution $\xi = (1-cw)/cu$ in (2), we obtain the form

$$y = sgn(c)\left[\frac{1}{cu}\int_0^x \frac{\xi dx}{\sqrt{(\xi^2-1)}} - \int_0^x \sqrt{(\xi^2-1)}dx\right]. \tag{4}$$

For the current function, $w = 2-x$, $dx = ud\xi$, and (4) is readily integrated into

$$y = sgn(c)\cdot\frac{u}{2}\left[\cosh^{-1}\xi + \left(\frac{2}{cu} - \xi\right)\sqrt{(\xi^2-1)}\right]_{\xi_0}^{\xi}.$$

From this, the curves in Fig. 2 were drawn for the values $u = 3$, and c between the limits $-\frac{1}{2} \leqq c \leqq \frac{1}{5}$. This gives the extremal curves only for positive values of x, the other half being drawn from symmetry. Other interesting sets of curves can be obtained by taking u so that $w_0 > u > w_1$.

This problem suggests at once the extension to the case of a curved stream in which both the magnitude and direction of the current would be a function of the two variables x and y. Here we could also consider the problem with a variable end point, viz., starting from a fixed point on one bank, to determine

[1] M. Merriman, *Treatise on Hydraulics*, 10th edition, p. 321.

the course of the boat so that the opposite bank could be reached in minimum time, with no specified landing point. Going into three dimensions, we could consider the course of minimum time for an airplane between two points. The Euler equations for the extremal curves would be more difficult to handle in general, but might be simplified for certain special current functions. The vector calculus should prove useful in this connection.

SOME CURIOUS GEOMETRIC PROPERTIES CONNECTED WITH POLYGENIC FUNCTIONS

By FREDERIC H. MILLER, Columbia University

In this note we consider some geometric relations among certain direction angles in the z-, γ- and σ-planes belonging to polygenic functions[1] and their first and second derivatives. By a polygenic function (in contrast with a monogenic function) of the complex variable $z=x+iy$ is meant any function of the form

$$w = \phi(x,y) + i\psi(x,y) \tag{1}$$

for which the Cauchy-Riemann equations are not obeyed. Here we assume that for some region the components ϕ and ψ of w and their first and second derivatives are continuous real functions of the real variables x and y. The derivative dw/dz then depends not only on the value of z but also on the slope $y'=dy/dx$; thus, we have:[2]

$$\frac{dw}{dz} = \gamma = \alpha + i\beta = \frac{w_x + w_y y'}{1 + iy'}. \tag{2}$$

The quantity γ therefore has infinitely many complex values at a point. Differentiation of (2) leads to the second total derivative of w with respect to z:[3]

$$\frac{d^2w}{dz^2} = \sigma = \xi + i\eta = \frac{w_{xx} + 2w_{xy}y' + w_{yy}y'^2}{(1 + iy')^2} + \frac{w_y - iw_x}{(1 + iy')^3}y''. \tag{3}$$

Evidently σ is a function of z and of the differential element (y', y'') of the second order along which the point is approached; that is, it depends on both the direction of approach and the curvature, as well as on the point in the

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[1] So named and extensively studied by Edward Kasner in a series of papers, among which the following are hereinafter referred to by number: I. *A new theory of polygenic (or non-monogenic) functions*, Science, vol. 66 (1927), pp. 581–582; II. *General theory of polygenic or non-monogenic functions*. The derivative congruence of circles, Proceedings of the National Academy of Science, vol. 14 (1928), pp. 75–82; III. *The second derivative of a polygenic function*, Transactions of the American Mathematical Society, vol. 30 (1928), pp. 803–818.

[2] II, p. 77.

[3] III, p. 804.

z-plane. Hence d^2w/dz^2 takes on ∞^2 values for each value of z, the derivatives y' and y'' varying simultaneously. But since the expression for σ is an integral linear function of y'', it is seen that for a fixed value of y' the locus in the σ-plane will be a straight line.

If we replace the partial derivatives of w in equations (2) and (3) by their values from (1) and separate into real and imaginary components, we obtain equations for α and β involving y' and equations for ξ and η containing both y' and y''. Elimination of y' from the former pair of equations leads to the equation of the derivative circle,[4]

$$(\alpha - H)^2 + (\beta - K)^2 = h^2 + k^2 = R^2, \tag{4}$$

the locus of points γ corresponding to a given point z. Here

$$\begin{aligned} 2H &= \phi_x + \psi_y, & 2K &= -\phi_y + \psi_x, \\ 2h &= \phi_x - \psi_y, & 2k &= \phi_y + \psi_x. \end{aligned} \tag{5}$$

Letting θ_z denote the direction angle of the element in the plane of $z=x+iy$, so that $\tan\theta_z = y' = m$, it is easily found that the inclination θ_γ of the vector from the center (H, K) of the derivative circle to a point on it is given by

$$\tan\theta_\gamma = \frac{\beta - K}{\alpha - H} = \frac{km^2 + 2hm - k}{hm^2 - 2km - h}. \tag{6}$$

Likewise, elimination of y'' from the equations for ξ and η secures us the equation of the σ-line,[5] whose slope is

$$\tan\theta_\sigma = \frac{km^3 + 3hm^2 - 3km - h}{hm^3 - 3km^2 - 3hm + k}. \tag{7}$$

Now it may be readily verified that a simple relation connects these three direction angles; we have, in fact,

$$\tan(\theta_z - \theta_\gamma) = \cot\theta_\sigma. \tag{8}$$

Assigning merely direction but not a sense to the three elements in their respective planes, we may restrict the direction angles to values $0° \leqq \theta < 180°$. With this understanding, it is clear that

$$\theta_z + \theta_\sigma = \theta_\gamma + (2n + 1)90°, \tag{9}$$

where n must have one of the values -1, 0, 1. It is of particular interest to note that this interdependence among the three angles is not affected by the function w or the values assumed by z.

We proceed to the implications of equation (9). For better comparison of

[4] I, p. 581.
[5] III, p. 812.

angular relations, let us suppose the three planes superimposed; then the following geometric properties are evinced:

(a) If the z-element is horizontal, the γ-radius will be perpendicular to the σ-line. The γ-vector then assumes the position of the phase vector[6] $h+ik$; thus, if we denote by $\tan\theta_c$ the ratio k/h (the slope of the phase vector), we now have for $\theta_z=0°$, $\theta_\gamma=\theta_c$ and $\theta_\sigma=\theta_c+(2n+1)90°$.

(b) If the z-element is in a vertical position, equation (9) tells us that the γ-radius and σ-line are parallel. Since the vector in the γ-plane moves through twice the angle traversed by the z-element and in the opposite direction,[7] we have $\theta_\gamma=\theta_\sigma=\theta_c$ when $\theta_z=90°$.

(c) If the element in the z-plane is parallel to the radius in the γ-plane, the σ-line is vertical.

(d) If the z-element and the γ-radius are at right angles, the σ-line is horizontal.

(e) Suppose now that θ_z is increased from any particular value by an amount δ. Then θ_γ will decrease by 2δ, and equation (9) shows[8] immediately that θ_σ must be lessened by 3δ. Thus the three inclinations change in the ratio $1:-2:-3$.

(f) Due to this ratio of rotation, the z-element and σ-line will be parallel when $\theta_z=\theta_\sigma=[\theta_c+(2m+1)90°]/4$, where m is some integer. Combining this with (9), we find

$$(2m-4n-1)90° - \theta_\gamma = \theta_\gamma - \theta_c\ ; \tag{10}$$

hence, when the z-element and σ-line are parallel, the γ-radius will bisect one of the angles between the phase vector $h+ik$ and the vertical.

(g) By rotating the z-element an odd-numbered multiple of 22.5° from the position of (f), it will become perpendicular to the σ-line. Then, for p some integer, we get from (9) and (10), $\theta_z-\theta_\sigma=(2p+1)90°$ and

$$(m-2n-p-1)180° - \theta_\gamma = \theta_\gamma - \theta_c, \tag{11}$$

whence we see that the γ-radius now bisects one of the angles between the phase vector and the horizontal.

(h) When the γ-vector is in a horizontal position, the z-element and σ-line are symmetrically situated with respect to the 45° line.

(i) Finally, if the γ-radius is vertical, the element in the z-plane and the σ-line are symmetrically situated with respect to the vertical.

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[6] I, p. 582.

[7] II, p. 80.

[8] For this result, otherwise obtained, see also III, p. 812.

DOUBLY HOMOGENEOUS FUNCTIONAL EQUATIONS[1]

By JOSEPH D. GRANT, James Millikin University

An algebraic addition theorem[2] is defined as a polynomial relation between $F(x)$, $F(y)$, and $F(x+y)$. This may be generalized to a polynomial relation between the functions of various linear homogeneous arguments which we shall call a homogeneous functional equation. The purpose of this paper is to show that, if such equations have constant coefficients and may be normalized, the solution may be made to depend upon the solution of an ordinary differential equation.

Theorems III and IV of the first paper hold and Theorem II becomes: *A solution $F(x)$ implies a solution $F(mx)$.* Theorem I takes the following form for this generalized situation:

Theorem V. *A necessary condition that a homogeneous functional equation have a solution which is analytic at zero*

(a) *except for a pole of order $v>0$ is that the terms of highest degree form an equation having the solution $A_{-v}x^{-v}$.*

(b) *and has there a zero of order $v>0$ is that the terms of lowest degree form an equation having the solution $A_v x^v$.*

(c) *and has there the finite value $k\neq 0$ is that k be a root in the equation in $F(0)$ formed by making all the variables zero, and that the linear terms resulting from the substitution $F(x)=k+G(x)$ form an equation having a solution $A_v x^v$, $v>0$.*

In part (c), k is a solution in any case, but will be trivial if it stands alone without a second term in the series. The test for a second term given in part (c) applies equally well to parts (a) and (b) except that it leads to the consideration of equations whose coefficients are not constants.

As examples we may consider, in order, the usual equations for cotangent, tangent, and cosine

$$V(\text{a}): \qquad F(x)F(y) - F(x)F(x+y) - F(y)F(x+y) = 1$$

for which

$$F(x)F(y) - F(x)F(x+y) - F(y)F(x+y) = 0$$

has the solution $A_{-1}x^{-1}$.

$$V(\text{b}): \qquad F(x+y) - F(x) - F(y) = F(x)F(y)F(x+y)$$

for which

$$F(x+y) - F(x) - F(y) = 0$$

has the solution $A_1 x$.

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[1] This is the author's second paper with this title. The first one was published in this Monthly, vol. 36 (1929), pp. 267–273.

[2] Forsyth's *Theory of Functions* (1918), Chapter 13; also Hancock's *Theory of Elliptic Functions* (1910), chapter 2.

$V(c)$: $\quad F(x+y)+F(x-y)=2F(x)F(y)$

for which

$$x=y=0, \quad F(0)=k\neq 0, \quad 2(k^2-k)=0, \quad k=1$$

and

$$G(x+y)+G(x-y)-2G(x)-2G(y)=0$$

has the solution A_2x^2.

As for Theorem I, the proof follows from a consideration of the identities which arise upon the substitution of a power series with undetermined coefficients.

That the homogeneous equations define the same range of functions as those which are doubly homogeneous is given by

Theorem VI. *For every homogeneous functional equation there exists a doubly homogeneous equation which includes among its solutions all those of the homogeneous equation.*

To prove this it will suffice to show an example of the method of formation of the doubly homogeneous equation and to note that the method is general. The following equation is satisfied by the Weierstrass "P" function:[3]

$$[F(x+y)+F(x-y)][F(x)-F(y)]^2 - 2F(x)F(y)[F(x)+F(y)]+\tfrac{1}{2}g_2[F(x)+F(y)]+g_3=0.$$

From this equation form two more having the same solutions by replacing x by $x+y$ and by $x-y$. The condition that these three equations be consistent in $\frac{1}{2}g_2$ and g_3 is the vanishing of the following determinant:

$$\begin{vmatrix} [F(x)+F(x-2y)][F(x-y)-F(y)]^2-2F(x-y)F(y)[F(x-y)+F(y)], & F(x-y)+F(y), & 1 \\ [F(x+y)+F(x-y)][F(x)-F(y)]^2-2F(x)F(y)[F(x)+F(y)], & F(x)+F(y), & 1 \\ [F(x+2y)+F(x)][F(x+y)-F(y)]^2-2F(x+y)F(y)[F(x+y)+F(y)], & F(x+y)+F(y), & 1 \end{vmatrix}$$

The result is clearly doubly homogeneous and is satisfied by any solution of the original. In general, one groups together terms of like degree, writes by repeated substitutions as many equations as there are different degrees for the terms, and eliminates the coefficients of the groups of terms to obtain the required doubly homogeneous equation.

If some variable, say x, enters each of the arguments with the same coefficient which may be taken as unity the equation will be said to be in normal form. It is possible to normalize some equations as is shown by the following example: $F(x+5y)F(2x)-F(x-3y)F(2x+4y)=0$ has solution e^{nx^2}.

Form a second equation by replacing x by $x+2y$ and a third by replacing y by $2y$. The condition that these three equations be consistent in $F(2x)$, $F(2x+4y)$, and $-F(2x+8y)$ is

[3] Whittaker and Watson's *Modern Analysis*, 3rd edition, p. 442.

$$\begin{vmatrix} F(x+5y), & -F(x-3y), & 0 \\ 0, & F(x+7y), & F(x-y) \\ F(x+10y), & 0, & F(x-6y) \end{vmatrix} = 0.$$

The resulting equation is clearly in normal form and has the solution $ke^{mx}e^{nx^2}$. In general if the function of arguments requiring elimination enter linearly the above process will suffice. The remarkable simplicity of normal form equations is accounted for, in part at least, by the following theorem.

Theorem VII. *For normal form equations, a solution $F(x)$ implies a solution $F(x+a)$.*

To prove this one may replace x by $x+a$ and then $F(z+a)$ by $G(z)$. $G(x)$ is seen to satisfy the same equation as $F(x)$, which proves the theorem.

Several interesting corollaries appear when this theorem is considered in connection with theorem one which stated that the first term of an ascending power series representing a solution must be a monomial solution. If $F(x+a)$ be expanded in Taylor's series in powers of x, the first term will be $F(a)$. Since $F(a)$ will not in general be zero or infinite, an equation having a solution will have an arbitrary constant solution. If $F(x)$ has a zero or pole of order v, in the finite plane there will be some value of "a" which will place it at zero so that the first term of the power series, and hence a monomial solution, will be x^v or x^{-v}. Since a doubly periodic function must have poles in the finite plane,[4] a necessary condition for the existence of doubly periodic solutions is that there be a solution $x^{-v}(v>0)$. One also notes that complementary trigonometric functions will appear as solutions of the same equations.

It will be convenient to give the method of solution in detail for the general second degree equation in normal form in two variables which is

$$\sum_{i=1}^{n} c_i F(x+b_i y)F(x+d_i y) = 0. \tag{1}$$

The solution is effected by means of related differential equations. Expand each function in equation (1) by Taylor's series in powers of y and arrange the result in ascending powers of y to obtain

$$\sum_{k=0}^{\infty} \phi_k y^k = 0 \tag{2}$$

wherein

$$\phi_0 = (0,0)Z^2,\ \phi_1 = (0,1)ZZ',\ \phi_k = \sum_{j=0}^{t} (j, i-j)\frac{Z^{(j)}Z^{(i-j)}}{j!(i-j)!}, \tag{3}$$

$$(p,q) = \sum_{i=1}^{n} c_i(b_i^p b_i^q + b_i^q d_i^p),\ (p,p) = \sum_{i=1}^{n} c_i b_i^p d_i^p. \tag{4}$$

$Z^{(0)} = F(x)$, t is $\frac{1}{2}k+1$ if k is even and $\frac{1}{2}(k+1)$ if k is odd.

[4] Whittaker and Watson's *Modern Analysis*, 3rd edition, p. 432.

Theorem VIII. *A necessary and sufficient condition that $F(x)$ satisfy equation* (1) *is that it satisfy each of the differential equations $\phi_k=0$.*

The proof of this follows from the equivalence of the series (2) and the equation (1) and the fact that both must be satisfied identically in y by a solution.

To obtain the solution of the functional equation it is not (of course) necessary to solve the infinite set of equations $\phi_k=0$. In any particular case one finds $\phi_k\equiv 0$, $k=0, 1, \cdots, m-1$ and $\phi_m\not\equiv 0$, that is, some coefficient in ϕ_m is not zero. Differential equations of this type have been shown always to possess solutions analytic at zero which involve as many arbitrary parameters as the order of the highest derivative which appears in the equation.[5] The solutions of $\phi_m=0$ must then be substituted in the functional equation as they will not in general satisfy all the equations $\phi_k=0$.

The essential properties of normal form equations, such as solutions and form, are invariant under the transformation

$$x = x' + \alpha y' \; ; \; y = \beta y'.$$

Before carrying out the computations for any particular example, it is desirable to have the equation in as simple a form as possible.

Theorem IX. *For every normal form equation, there exists one of the same degree having the same ϕ_m which is invariant under the transformation $y=-y'$.*

To prove this change y to $-y$ in equation (1) and add the result to equation (1) if m is even or subtract if m is odd. The resulting equation will then satisfy the conditions of the theorem.

The normal form equations in two variables thus separate into two types which may be designated as even and odd from their analogy to the equations $F(y)-F(-y)=0$ and $F(y)+F(-y)=0$, respectively. The general even equation of second degree is

$$\sum_{i=1}^{n} c_i\{F(x+b_iy)F(x+d_iy) - F(x-b_iy)F(x-d_iy)\} = 0.$$

$$(p,q) = \sum_{i=1}^{n} c_i\{b_i^p d_i^q + b_i^q d_i^p - (-b_i)^p(-d_i)^q - (-b_i)^q(-d_i)^p\}, \quad (p,p)\equiv 0.$$

One notices that $\phi_{2r}\equiv 0$ for all r's.

Similarly for the general odd equation of second degree,

$$\sum_{i=1}^{n} c_i\{F(x+b_iy)F(x+d_iy) + F(x-b_iy)F(x-d_iy)\} = 0.$$

One notices that $\phi_{2r+1}\equiv 0$ for all r's. As a special subclass of the odd equations one has those in which every term is invariant.

To illustrate the method we shall solve the following even equation:

[5] Forsyth's *Differential Equations* (1900), vol. 2, p. 26.

$$
\begin{aligned}
&F(x-y)F^2(x)F(x+y)F(x+2y) - F(x-2y)F(x-y)F^2(x)F(x+y)\\
&\quad + F(x-2y)F^3(x-y)F(x+y) - F(x-y)F^3(x+y)F(x+2y)\\
&\quad + F(x-2y)F(x)F^2(x+y)F(x+2y) - F(x-2y)F^2(x-y)F(x)F(x+2y)\\
&\quad + F^3(x-y)F(x)F(x+y) - F(x-y)F(x)F^3(x+y)\\
&\quad + F^2(x)F^2(x+y)F(x+2y) - F(x-2y)F^2(x-y)F^2(x)\\
&\quad + F(x+2y)F^2(x-y)F^2(x+y) - F^2(x-y)F^2(x+y)F(x+2y) = 0.
\end{aligned}
\tag{5}
$$

One sees that a solution $F(x)$ implies a solution $[F(x)]^{-1}$. For, if one replaces $F(x)$ by $1/G(x)$ and multiplies throughout by $G(x-2y)G^3(x-y)G^2(x)G^3(x+y)G(x+2y)$, $G(x)$ is seen to satisfy the same equation as $F(x)$.

$$\phi_m = Z(Z^2Z'Z^{(4)} - Z^2Z''Z''' - 3ZZ'^2Z''' + 3Z'^3Z'') = 0.$$

By three quadratures this is reduced to $Z'^2 = B_0 + B_2Z^2 + B_4Z^4$ whose general solution[6] is $c \cdot sn(ax+b, k)$. There are also a number of degenerate solutions, some of which are

$c \cdot \sin(ax+b)$, $B_4 = 0$; $c \cdot \tan(ax+b)$, $B_2^2 = 4B_0B_4$; $c \cdot \csc(ax+b)$, $B_0 = 0$;

$c \cdot e^{ax}$, $B_0 = B_4 = 0$; $(ax+b)$, $B_2 = B_4 = 0$; $(ax+b)^{-1}$, $B_0 = B_2 = 0$.

To complete the solution we must show that $sn(x, k)$ is a solution of equation (2). Pierce's integral tables give, as number 725,

$$sn(x+y)\,sn(x-y)\{1 - k^2\,sn^2 x\,sn^2 y\} = sn^2 x - sn^2 y.$$

From this form two equations by replacing x by $x+y$ and by $x-y$. The quantities k^2sn^2y and sn^2y enter linearly into the three equations and their elimination results in equation (5).

As an example of an odd equation we shall solve

$$
\begin{aligned}
&F^2(x+y)F(x-2y) + F^2(x-y)F(x+2y)\\
&\qquad + F^3(x) - F(x+2y)F(x)F(x-2y) - 2F(x+y)F(x)F(x-y) = 0.
\end{aligned}
$$

A solution $F(x)$ implies a solution $e^{nx}F(x)$, by Theorem III:

$$\phi_m = ZZ''Z^{(4)} - Z'^2Z^{(4)} - ZZ'''^2 + 2Z'Z''Z''' - Z''^3 = 0.$$

Two quadratures will put this in the form

$$Z'' - 2nZ' + (n^2 + a^2)Z = 0,$$

whose general solution is $c \cdot e^{nx}\sin(ax+b)$. One easily checks the solution $\sin x$.

However, it is not always true that every solution of $\phi_m = 0$ is a solution of the functional equation. To see this we may consider the following odd equation which is symmetric term by term:

---

[6] $sn(x+b, k)$ is defined as the solution of $z'^2 = (1-z^2)(1-k^2z^2)$.

$$F(x+2y)F(x-2y) - 4F(x+y)F(x-y) + 3F^2(x) = 0. \tag{6}$$

$$\phi_m = ZZ^{(4)} - 4Z'Z''' + 3Z''^2 = 0,$$

which may be written

$$Z^2\left\{\frac{d^4}{dx^4}(\log Z) + 6\left[\frac{d^2}{dx^2}(\log Z)\right]^2\right\} = 0,$$

whose general solution is seen to be $c \cdot e^{nx}\sigma(x+b, 0, g_3)$. To check the solution $\sigma(x, 0, g_3)$, one would need to know properties in addition to its differential equation. If it is not a solution, it may be easier to find one of the $\phi_k = 0$ which it does not satisfy. In this case $\sigma(x, 0, g_3)$ is not a solution of

$$\phi_6 = ZZ^{(6)} - 6Z'Z^{(5)} + 15Z''Z^{(4)} - 10Z'''^2 = 0.$$

The general solution of (6) will then be seen to be the degenerate solution of $\phi_m = 0$, namely $e^{nx}(ax+b)$.

It will be of interest as a further illustration to determine all the odd second degree normal form equations in three arguments which can exist. The general equation is

$$\begin{aligned} &c_0F^2(x)+c_1F(x+y)F(x-y)+c_2[F^2(x+y)+F^2(x-y)] \\ &\quad +c_3F(x)[F(x+y)+F(x-y)]=0. \end{aligned} \tag{7}$$

A necessary condition that there be any solution analytic at zero is $(0, 0) = c_0+c_1+2c_2+2c_3=0$. The other coefficients one needs to consider are

$$\begin{aligned} &(0,2) = 2c_1 + 4c_2 + 2c_3\,; && (1,3) = 2(1,1)\,; && (1,5) = 2(1,1)\,; \\ &(1,1) = -c_1 + 2c_2\,; && (2,2) = c_1 + 2c_2\,; && (2,4) = 2(2,2)\,; \\ &(0,4) = (0,2)\,; && (0,6) = (0,2)\,; && (3,3) = (1,1). \end{aligned}$$

If the equation (7) have factors rational in F, the problem is not one of second degree equations and may be omitted from consideration. The condition for such factors is

$$\Delta = \begin{vmatrix} 2c_2 & c_1 & c_3 \\ c_1 & 2c_2 & c_3 \\ c_3 & c_3 & 2c_0 \end{vmatrix} = 0.$$

Substituting the value of c_0 from $(0, 0) = 0$, one finds

$$\Delta = (c_1 - 2c_2)(c_1 + 2c_2 + c_3)^2 = 0,$$

$$\Delta = (1,1)(0,2)^2 = 0.$$

From this one sees that the necessary and sufficient condition that the equation (7) shall not have factors rational in F is that neither (0, 2) nor (1, 1) be zero.

Let us write, then, $(0, 2) = a \neq 0$; $2(1, 1) = b \neq 0$; $6(2, 2) = n$.

The differential equations having the coefficients are

$$\begin{aligned} \phi_2 &= aZZ'' + bZ'^2 = 0, \\ \phi_4 &= aZZ^{(4)} + 4bZ'Z''' + nZ''^2 = 0, \\ \phi_6 &= aZZ^{(6)} + 4bZ'Z^{(5)} + 5nZ''Z^{(4)} + 10bZ'''^2 = 0. \end{aligned} \tag{8}$$

Since any solution of equation (7) is a solution of each of the equations (8), it is necessary that the equations (8) be consistent.

The consistency condition for $\phi_2 = 0$, $\phi_2' = 0$, $\phi_2'' = 0$, $\phi_4 = 0$ is

$$nb = (2a - b)(a + 2b). \tag{9}$$

The consistency condition for $\phi_2 = 0$, $\phi_2' = 0$, $\phi_2'' = 0$, $\phi_2''' = 0$, $\phi_2^{(4)} = 0$, $\phi_6 = 0$, after writing for n its value from (9), is

$$(a - b)(a + b)(a + 2b)(2a + b) = 0.$$

Hence there can not be more than four odd second degree normal form equations in three arguments which do not have rational factors in F. The four equations exist and are unique. The following list then comprises all possibilities.

I. $a - b = 0$, $\quad ZZ'' + Z'^2 = 0$, $\quad Z = (px + q)^{1/2}$.

$$F^2(x + y) + F^2(x - y) - 2F^2(x) = 0.$$

II. $a + b = 0$, $\quad ZZ'' - Z'^2 = 0$, $\quad Z = qe^{px}$.

$$F(x + y)F(x - y) - F^2(x) = 0.$$

III. $2a + b = 0$, $\quad ZZ'' - 2Z'^2 = 0$, $\quad Z = (px + q)^{-1}$.

$$2F(x + y)F(x - y) - F(x)F(x + y) - F(x)F(x - y) = 0.$$

IV. $a + 2b = 0$, $\quad 2ZZ'' - Z'^2 = 0$, $\quad Z = (px + q)^2$.

$$F^2(x + y) + F^2(x - y) - 2F(x + y)F(x - y) - 16F^2(x) + 8F(x)F(x - y) + 8F(x)F(x + y) = 0.$$

The problem of exponential multipliers for solutions of normal form equations is easily handled. Before stating the theorem it will be desirable to show the existence of an equation for which a solution $F(x)$ implies a solution $e^{nz^v}F(x)$ for each positive integer v. For equations in one variable this is seen from $F(2x) = F^2(x)$. The two term normal form equation

$$\prod_{i=1}^{r} F(x + a_i y) = \prod_{i=1}^{r} F(x + b_i y)$$

will have this property if[7] $a_i \overset{v}{=} b_i$. A few sets of such numbers will be of interest

$$-1, +1 \overset{1}{=} 0,0,$$
$$-1, -1, +2 \overset{2}{=} +1, +1, -2$$
$$-3, -4, +3, +4 \overset{3}{=} -5, +5, 0, 0,$$
$$-1, -5, -9, +5, +8 \overset{4}{=} -7, -8, 1, 5, 9,$$
$$-5, -6, -11, +5, +6, +11 \overset{5}{=} -1, -9, -10, +1, +9, +10$$

That sets will exist for all positive integers v is shown by a theorem due to E. B. Escott.[8]

$a_i \overset{v}{=} b_i$, $k \neq 0$ implies a_i, $b_i + k \overset{v+1}{=} a_i + k$, b_i.

For the two term equation

$$\phi_m = Z^r \frac{d^{v+1}}{dx^{v+1}}(\log Z) = 0.$$

Theorem X: *The necessary and sufficient condition that for the equation*

$$\sum_{j=1}^{n} c_j \prod_{i=1}^{r} F(x + b_{ij}y) = 0$$

a solution $F(x)$ *imply a solution* $e^{xv}F(x)$ *is that* $\sum_{i=1}^{r} b_{ij}^{w} = M_w$ (*independent of* j) *for each* $w \leqq v$.

The proof follows by direct substitution.

Two corollaries follow at once. Since the conditions are symmetric functions of the b_{ij}, $v+1 \leqq r$. Also a multiplier e^{xv} implies a multiplier e^{xv-1}.

The following equation for the sigma function will further illustrate the theorem

$$F(x+3y)F(x+y)F^4(x-y) + F(x+2y)F(x+y)F^3(x)F(x-3y)$$
$$+ F(x+2y)F^2(x+y)F(x)F^2(x-2y) - F(x-3x)F(x-y)F^4(x+y)$$
$$- F(x-2y)F(x-y)F^3(x)F(x+3y) - F(x-2y)F^2(x-y)F(x)F^2(x+2y)$$
$$= 0.$$

The sum of b_{ij} is zero for each term and the sum of b_{ij}^2 for each term is 14. The solution is $ke^{nxe^{mx^2}}\sigma(x+b, g_2, g_3)$.

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[7] L. E. Dickson's *History of the Theory of Numbers*, vol. 2, chapter 24. The symbol $\overset{v}{=}$ means that the sums of all positive powers of the two sets of integers are equal up to and including the power v.

[8] *Quarterly Journal of Pure and Applied Mathematics*, vol. 41 (1910), p. 144.

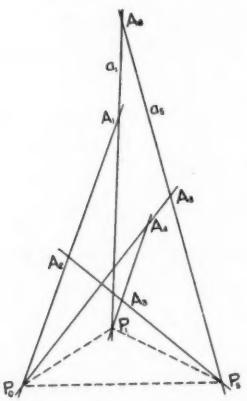
GENERALIZATIONS OF PASCAL'S AND BRIANCHON'S THEOREMS

By GLENN JAMES, University of California at Los Angeles

For convenience we first consider the converse of Pascal's theorem and state it in the following form: *If the intersections of opposite sides of a hexagon are always collinear any vertex describes the conic determined by the other five.* We shall consider five of the vertices, hence four of the sides, as fixed. One pair of opposite sides will then be fixed and their intersection will be the base of a pencil described by the line on the intersections of the other two pairs of opposite sides. The base of this pencil will, of course, be the center of perspectivity of the two ranges cut off by the pencil on the other two fixed sides. We shall speak of this pencil and any ranges on these fixed sides, whether or not they are perspective, as the Pascal pencil and the Pascal ranges. The varying vertex of the hexagon will be said to be opposite the line joining corresponding points on the Pascal ranges. The object of this paper is to determine the loci of the varying vertex when the intersections of opposite sides of the hexagon are not necessarily collinear but certain restrictions have been placed upon the triangle which they form. Obviously, the Pascal ranges may then be projective and not perspective and all six vertices of the hexagon may not lie on the same conic. The conic determined by the five fixed vertices and the triangle formed by the intersections of opposite sides of the hexagon will be termed respectively the base conic and the defining triangle.

Theorem I: *If the line on corresponding points of the Pascal ranges envelopes a conic tangent to the bases of the ranges, the opposite vertex of the hexagon describes a conic which has the two neighboring vertices in common with the base conic, and conversely.*

Proof: Denote the fixed vertices of the hexagon by A_1, A_2, A_3, A_4, A_5,



the varying vertex by A_6 and the Pascal ranges on the sides opposite A_1A_6 and A_5A_6, respectively by P_1 and P_5. Then the pencils A_1A_6 and A_5A_6 are projective whenever the ranges P_1 and P_2 are since the ranges are sections of the pencils.

This simple theorem includes the Pascal theorem and its converse; for when the ranges P_1 and P_2 are perspective A_3 is a self-corresponding point and $A_1A_2-A_4A_5$ is the center of perspectivity. The dual of this theorem bears a similar relation to Brianchon's theorem.

Theorem II: *If the determinant* $|X_1, Y_2, 1|$, *where* X_iY_i *are the cartesian coordinates of the intersections of opposite sides of a hexagon is equal to a constant then any vertex describes a conic which cuts its base conic in four real points, two on each of the moving sides when these are parallel to their opposite sides, and conversely.*

Proof: Since the coordinates of the moving point enter linearly[1] in the coefficients of the equations of the moving sides, the equation of the locus of the moving point is of the form

$$\Delta = KD_1D_2D_3, \tag{1}$$

where K is some constant, $\Delta=0$ is the equation of the base conic, D_1 is the determinant of the equations of the pair of fixed opposite sides and D_2 and D_3 are respectively the same functions of the other two pairs of opposite sides. The determinants D_2 and D_3 are linear in the coordinates of the moving point. Since (1) is quadratic, its locus obviously passes through all points on the base conic whose coordinates make either D_2 or D_3 zero. When the moving vertex coincides with a neighboring vertex of the hexagon, one of the moving sides is indeterminate. Hence D_2 or D_3, as the case may be, is zero. Each of these determinants is also zero when the varying line, whose equation enters into it, is parallel to its opposite side. To prove the converse, it suffices to note that any fifth point on the given conic, other than the four given points, will determine a real value of K.

It is of interest to note that when the moving point is on an intersection of the two conics the defining triangle, with an area not necessarily zero, reduces to an infinite segment of a straight line. In other words the area has taken the form $0 \cdot \infty$, which is defined to have the value chosen for the area of the triangle. In stating the above theorem, it has been assumed that the vertices of the defining triangle would be taken in a given order. If, however, we merely take the numerical value of the area of this triangle, (1) becomes a quartic whose locus is two conics, both of which are given by the following construction.

To construct points on the conic defined by (1), which we shall call a principal conic, we proceed as follows: Using the notation of Theorem I,

[1] Because of this relation, it is evident that one can impose many different conditions upon the defining triangle which will restrict A_6 to lie upon a conic. However, the one used in this theorem is typical and seems to be the most interesting.

draw through one of the fixed vertices, say A_5, an arbitrary line. Join the intersection of this line and A_2A_3 to the point $A_2A_3-A_4A_5$. Divide this line segment into twice the constant given for the area of the defining triangle. Then construct parallels to the line on the line segment at distances numerically equal to this quotient. The points where these parallels intersect A_3A_4, taken in turn with A_1 determine lines which intersect the arbitrary line through A_5 in two points, one on each of the principal conics defined by taking the vertices of the defining triangle in different orders.

It follows from equation (1) that the configuration of twelve conics (or twenty-four if we disregard algebraic sign) defined by Theorem 2 reduces to a single conic when the area of the defining triangle is zero. The dual of this theorem, that is the corresponding generalization of Brianchon's Theorem, is as follows:

Theorem 2. *If the determinant* $|A_1, B_2, 1|$, *where* $A_iX+B_iY+1=0$ *are the equations of the lines joining opposite vertices of a hexagon is equal to a constant, any side of the hexagon envelopes a conic which has four real tangents in common with its base conic, two on each of the moving points when these are ideal points, and conversely.*

QUESTIONS AND DISCUSSIONS

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSION

A Note on Horner's Method of Solving an Algebraic Equation

By C. W. Bruce, Wesleyan College, Macon, Ga.

In the determination of incommensurable roots of an algebraic equation by Horner's method after the roots have been isolated by some one of the various methods, the procedure as given in text books is as follows: Suppose one root lies between 2 and 3. First, the roots of the equation are diminished by the whole number 2. The resulting equation will have a root between 0 and 1. An approximate value of this root is found by neglecting the terms of second degree and higher, and setting the first degree term equal to the negative of the known term. The question arises: will this approximate value of the variable be an approximation to the desired root, or to some other root of the equation? There is no obvious reason why this should give one root rather than another. In answer to the above question let us consider the cubic $x^3+ax^2+bx+c=0$, which has incommensurable roots.

Case I. All the roots positive. Let the roots be r, r', and r'' where $r<r'<r''$ and r is not integral. Neglecting the first two terms of the equation we get

$$x = -c/b = rr'r''/(rr' + rr'' + r'r'').$$

If x is to be an approximation to r rather than to r' or r'', then must

$$\frac{rr'r''}{rr' + rr'' + r'r''} - r < r' - \frac{rr'r''}{rr' + rr'' + r'r''}.$$

Clearing of fractions and omitting terms which cancel gives $-r^2r' - rr'' < rr'^2 + r'^2r''$ which is evidently true, showing that when all the roots are positive the method gives an approximation to the smallest root.

Case II. One root negative, two positive. Let the roots be r, r', and r'', where $r<r'<r''$ and r' is not integral.

As in case I $x=rr'r''/(rr'+rr''+r'r'')$. The numerator is here negative. The two terms rr' and rr'' in the denominator are negative. The sign of x will be positive if $|rr'+rr''|>r'r''$. This may or may not be the case. Hence the usual process may or may not lead to the desired result in this case.

Case III. Two roots negative, one positive. Let the roots be r, r', and r'' where $r<r'<r''$ and r'' is not integral.

Again $x=rr'r''/(rr'+rr''+r'r'')$. The numerator here is positive. In the denominator rr' is positive, $r'r''+rr''$ is negative. x will be positive if $rr'>|r'r''+rr''|$.

If this inequality is not true the method again fails.

The method will always work for the smallest root provided the first transformation of the equation makes all the roots positive. The method may or may not work for the remaining roots. Since the numerical value of the root looked for is made small in the first transformation of the equation, the method works more often than it fails.

RECENT PUBLICATIONS

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All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

The Pastures of Wonder. By Cassius Jackson Keyser. Columbia University Press, New York, 1929. xii+208 pages. $2.75.

In this book the author has carried to the extreme the tendencies shown in his previous writings concerning mathematics, and has attached here also his views of science. He has departed widely from his view, once expressed, that science is "a sublimated form of play, the austere and lofty analogue of the kitten playing with the entangled skein or of the eaglet sporting with the mountain winds."

He discusses two "realms": that of mathematics, and that of science. That of mathematics, in brief, he asserts is nothing more than deductive logic. That of science he proposes to limit to the establishment of categorical pro-

positions. Mathematics is an edifice of hypotheticals, science an edifice of categoricals. With the first idea many will take issue. The proposal many will reject. That some mathematicians and some others have wished to limit mathematics to its purely logical aspect is of course true, but mathematicians generally refuse to agree that what is thus excluded is not mathematics. That the deductive reasonings from hypotheses in science do not constitute a legitimate part of science few will agree with. There are of course scientists who would limit science to accurate and ordered accounts of phenomena. But this limitation appeals to few. The author undertakes to meet certain challenges to his definitions, and of course disposes of them to his own satisfaction. The book is a natural outcome of the notions of the "logistic" development by Russell and others. The antidotes for this disease to be found in the writings of Poincaré and others need to be more widely spread. One of the most serious challenges to logistic is the simple question: What constructive mathematics has it ever produced? To confine mathematics to logistic is worse than to confine it to "the study of integers and what can be got out of them, and nothing else."

To quote an example which the author himself gives, the proposition that "the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides" is a categorical proposition, and is not therefore mathematical. This is, according to the author, the conclusion from a set of unstated "Euclidean" postulates, and a conclusion by itself is not a mathematical proposition. The mathematical proposition is the statement that this conclusion follows from the original postulates. On this basis all conclusions in mathematics would seem to be in science, not mathematics. The only thing left to mathematics is the barren statement that the conclusion has been "proved." The premises may be false, they may be mere empty symbols, and the conclusion may be nowhere applicable, but if the logic is correct (*and this is a big assumption*) then we have a mathematical statement. It is the old and absurd statement of Russell, that in mathematics we do not know what we are talking about nor if our conclusions are true. But one needs only to notice that we have struck here an endless sequence from which we cannot escape. For we should say above; *if the process of logic used in demonstrating Pythagoras' theorem is valid, then the conclusion is a consequence of the postulates.* Now this is itself a hypothetical proposition, and *its* conclusion is a *categorical proposition.* Thus the original statement that the theorem of Pythagoras follows from the postulates of Euclid is categorical and so not mathematical. This can be extended ad libitum. In fact, every hypothetical proposition is also a categorical proposition, and the distinction the author desires to make breaks down.

The statement of Benjamin Peirce, made years before Russell's work, that mathematics is the science that draws necessary conclusions, is the basis for these logistic arguments. But the other statement of Peirce is almost invariably omitted. It is that the processes of logic cannot be applied without being transmuted into various forms, and this transmutation is the mathematical process

in the inquiry. This is very important, for a large part of mathematical investigation does not consist in drawing conclusions, but in building structures of thinking. Most of mathematics consists in the development of ideal structures. The brilliant son of Benjamin Peirce, Charles Saunders Peirce, gave as the definition of mathematics, in the Century Dictionary, "the study of ideal constructions (often applicable to real problems), and the discovery thereby of relations between the parts of these constructions, before unknown." He adds, "The observations being upon objects of imagination merely, the discoveries of mathematics are susceptible of being rendered quite certain." Such observations are categorical assertions.

The author of the book under review makes much of the *forms* of propositions and elaborates this with various examples of syllogism. But he evidently fails to see that forms themselves are objects, and assertions about them are categorical assertions. It was Kempe who wished to define mathematics as the science of pure form, but he was quite clear in his vision of forms as ideal objects, in practically the same sense as C. S. Peirce. As the author says on page 99, mathematical verities are eternally true, but not because they are mere logic. Mere logic itself is an ideal creation of the human mind, and may be utterly changed. Witness the work of Brouwer on the law of excluded middle. They are true because the human mind has the ability to see the universal characters in its own creations.

As for the proposed definition of science, we may leave the scientist to accept or reject it. The discussion of the proposal with imaginary scientists, of course frames answers for the objections already set. The proposal seems to have as one aim to exclude mathematics from science. This has already been done if we accept the common modern view that science is based on the observation of the natural world. Also he tries to exclude the mathematical ideals of methodology and thought from science. This is being rapidly accomplished by modern physics in some directions, but is being equally rapidly shown to be impossible in others. The reviewer is referring to Bridgman's *Logic of Modern Physics*, and to all the present exposition of atoms as matrices. The author of course admits the necessity for the scientist to use mathematics, and probably would not deny the fact that many advances in mathematics have originated in scientific problems.

The title, "Pastures of Wonder," is ascribed in the preface to the existence of two great realms, the categorical and the possible (*equivalent to hypothetical?*) and the wonder these two create. But the two worlds are also called propositional, and one may well ask if the universe consists merely of propositions, for human experience would certainly deny this statement. And propositions later are defined to be only those statements which can be tagged as true or false. Are there no others of use? And the old, old question, which Russell gave up, still remains: What does true or false mean, and why choose one rather than the other?

The universe is indeed filled with pastures of wonder. One of these is the

pasture of beauty. One can defend the play of the mathematician as the loitering of an enchanted spirit in this pasture. And this pasture is not the rarefied ether, empty of all content. Mathematics would have died of inanition long ago were it that. For all creative mathematicians are chiefly concerned in making categorical assertions about the ideal world in which they live. This is what the "educated layman" to whom the author addresses his essay should understand as the meaning of the term mathematics. If the layman will pick up almost any book on mathematics he will find that the writer is chiefly concerned with the properties, the characters, the structure, the relations of the mathematical objects he is considering. And he should not be misled into thinking that the only feature of the book which is *mathematical* is the empty assertion that the conclusions follow from the premisses. He will find indeed that the greater part of the book is derived from direct intuition and not from logic at all. If he succeeds in seeing the structures discussed as the writer sees them he will be filled with wonder at their beauty.

JAMES BYRNIE SHAW

Elementary Differential Equations. By Thornton C. Fry. D. Van Nostrand Co., New York, 1929. x+255 pages.

A textbook on elementary differential equations consisting of 250 generously sized pages, written by one who is prominently connected with the mathematical research side of electrical engineering is sufficient to pique one's curiosity and justify the expectations that a book particularly suited to introducing engineers and engineering students to the desirability and fascination of differential equations has appeared. The book, as stated in the introduction, is actually based on out-of-hour courses of the Bell Telephone Laboratories. This fact is, of course, responsible for some of the idiosyncrasies of the book, more detailed explanation of some of the elementary processes, slighting of some topics, and overemphasis on others, especially linear differential equations with constant coefficients, and many fascinating side remarks, which enliven the text and the footnotes.

The amount of actual material in differential equations covered in the book is a little disappointing in its meagerness, when one considers the size of the book. Analysis shows that the book covers little more than one frequently finds in the closing chapters of a good first book on calculus: i.e., the standard methods for solving ordinary differential equations of the first order, a passing mention of the solvable cases of the second order, where dependent or independent variable are absent (treated as a parting shot in the chapter on differential equations of the first order), and linear differential equations with constant coefficients. The material not usually found in a book on calculus is a treatment of singular solutions of first order differential equations (why important for the engineer?) a rather extensive group of applications, the exposition of the symbolic method of solving linear differential equations with con-

stant coefficients, and systems of such, and a brief closing chapter on special linear differential equations of the second order, especially Bessel's equation.

To be commended in the book is the gradualness of the treatment, the slow steps leading up to the various methods, e.g., special preliminary consideration of changing of variable, and in many cases the elementary character of the exposition. Then there is the presentation of the symbolic method and principles of solving linear differential equations with constant coefficients, and the numerous examples chosen from the applications. Also one is delighted with the many and fascinating side remarks. For instance in discussing applications, we read: "The student will succeed best if he approaches the preliminary statements (of physical laws) with a judicious amount of faith, reserving his usual healthy skepticism for the mathematical arguments by which they are followed"; "The solution of differential equations is no fit occupation for the obtuse"; "Mathematical results have an almost annoyingly monotonous habit of being sensible"; "The more difficult problems, if they are to be of any practical significance, require the explanation of so much geometry, mechanics, or electricity as to consume a large amount of time that had better be devoted to mathematical ideas."

On the other hand, there are a number of points which might be viewed with disapproval. For instance, in the matter of terminology, the author is not always careful to use accepted mathematical terms. He confuses "algebraic" with "explicit" or "expressible in terms of elementary functions"; he frequently confuses order and degree of a differential equation; talks about allowing dx to vanish, instead of to approach zero; allows expressions to "degenerate" into derivatives; and so on. The treatment is a little uneven as to the plane on which it is kept. One finds advanced mathematical ideas playing tag with very elementary ones, especially in the symbolic section, and the jumps are sometimes very sudden. There is a lot of "sky" work, where only a deity or genius would suspect reasons for solving problems as worked out in the illustrations; and it rather worries one to find so many applications chosen from the mathematical field. For example, the author finds it necessary to expound the elements of the calculus of variations in a search for suitable differential equation applications. Are there not many nice problems available in the engineering field? The method of symbolic operators seems to be brought forth a little apologetically, and justified on the grounds that it gives the "correct answer." Any teacher bristles, when such a justification is brought forth by a student. After all the only justification for any mathematical operation or operator, or process is its consistency. If symbolic operations are desirable in the solutions of differential equations, why not define them accurately and then use them consistently. Thus the operation $1/(p-a)$ on a function $f(x)$ should be *defined* as the general solution of the linear differential equation $dy/dx-ay=f(x)$, and its properties deduced from the definition. In his exposition of the symbolic method the author is not always careful to draw a sharp line between his sympolic operators and numbers. It would contribute much to

avoid confusion, if entirely different letters were used for the two concepts. After all, what is the matter with D as a symbol for derivative, as contrasted with p?

The book should prove stimulating and helpful to the engineering student of differential equations, provided he is warned in advance that not all differential equations are linear with constant coefficients.

T. H. HILDEBRANDT

Seven Place Natural Trigonometric Functions, together with many miscellaneous tables and appendices on the adjustment of the engineer's transit and level, area computation, vertical curves, simple curves, and determination of latitude, longitude and azimuth. By Howard C. Ives. John Wiley and Sons, New York, 1929. vi+222 pages. $2.50.

The major part of this book consists of seven-place tables of the natural trigonometric functions. It is pointed out in the preface that the widespread adoption of calculating machines is having the effect of rendering logarithms to some extent obsolete, and necessitates the use of tables of natural functions.

Tables I–V give the six trigonometric functions, versed sines and exsecants, chords and co-chords, for angles differing by one minute. Table VI is the inevitable table of lengths of circular arcs with unit radius, that is to say a mere multiplication table for π; and table VII, transforming degrees to radians, merely repeats a part of VI. There is a table of powers, root, and reciprocals; and the remaining tables are chiefly of interest to the engineer.

An excellent syllabus of trigonometry, mensuration, and other topics needed by the engineer, together with a section on adjustments of instruments, should render this a most useful little volume.

To the mathematician the most obvious fault of the book is the lack of any suggestion for interpolation except by the usual method of first differences. In fact, in the explanation of the tables there are explicit directions for the use of this method. When we are dealing with seven places of decimals there is evidently danger that simple interpolation will not be accurate; the possible error is of course of the order of one-eighth of the second differences. It turns out that simple interpolation is accurate for all sines and cosines; and perhaps for all trigonometric functions less than unity. For tangents and secants of angles greater than 45°, however, correct results can be obtained only by using second differences, and directions for this process should accompany the tables.

R. A. J.

Plane Trigonometry (With Tables). By Ernest Jackson Oglesby and Hollis Raymond Cooley. Prentice-Hall, Inc., New York, 1929. xii+226 pages.

In the construction of this text the authors seem to have not only carefully considered the subject matter that should be included but also the best method of presenting and developing this subject matter that it might be attractive and interesting to the student as well as teachable for the teacher. One of the very

striking features of the book is the introduction of each new chapter with preliminary leading questions in order to put the student on the alert for the discussion which is to follow. The good effect of this scheme is enhanced by means of the occasional appearance of significant questions throughout the body of each chapter.

Additional features which the authors "think merit the special consideration of teachers of mathematics" are the introduction of "the trigonometric functions of the general angle before proceeding with the special application to the acute angle," the treatment of the general triangle, and the two last chapters of the book which are devoted to polar coordinates and complex numbers respectively. In the discussion of the general triangle the law of sines has been introduced and then the far-reaching application of this law and the need for other formulae are shown in the discussion of the four cases that arise in the solution of the general triangle. The addition formulae are introduced as a means by which the law of tangents may be derived, the purpose being to illustrate at least one practical application of these formulae.

The introduction to the use of logarithmic tables is very well developed and seems to present the necessary instruction in a carefully planned manner. The book also contains a rather complete and usable set of logarithmic and trigonometric tables.

Last, but by no means least, the printers have done an excellent job in the printing of this text. The type is such that the reading will not be a strain on one's eyes, the figures and tables are clear and well constructed, and a good grade of paper has been used.

In general, the text impresses one as being a book that deserves the careful examination of those teachers who desire texts that will not only contribute to the information of their students but also tend to develop their interest in mathematics as well as to cultivate an appreciation of mathematics.

F. L. WREN

Statistical Mechanics. By R. H. Fowler. Cambridge University Press, 1929. The Macmillan Company, New York. viii+570 pp. 35 shillings.

So long as a dynamical system has a reasonably small number of degrees of freedom, it is practical to study the evolution of the system by an examination or actual solution of the equations of motion. For dynamical systems of very many degrees of freedom, such as a gas, this detailed study is out of the question, and one utilizes the methods of statistical mechanics. If the number of degrees of freedom is, for example, n, a complexion of the system is completely specified by giving the location of a point in a so-called "phase space" of $2n$ dimensions, n of the variables corresponding to the positional coordinates $q_1 \cdots q_n$ of the dynamical system, while the remaining n variables correspond to the associated momental coordinates $p_1 \cdots p_n$. A single point in the phase space thus represents a system, and the trajectory of this single point gives

the successive configurations through which the system passes. It is, moreover, an essential part of the technique of statistical mechanics to consider not this representative point of a single system, but a whole dust of points in the phase space corresponding to a very large number or "ensemble" of systems, alike in their constitutions but differing in their phases.

Having assumed a certain distribution of such representative points in the phase space, one then seeks to determine what physical properties are enjoyed by an overwhelming majority of the dynamical systems which the various points represent. If a certain property is characteristic of all but an infinitesimal[1] fraction of the systems, one then concludes that this property will be possessed by one actual system during all but a negligibly small fraction of the time. There are two subtle difficulties here, neither of which has ever been satisfactorily analyzed; viz., there is no really satisfactory basis for the choice of density of the representative points in the phase space, and there is no satisfactory justification for the identification of the number average over the ensemble with the time average for one system.

From a mathematical point of view, the two matters just mentioned are among the most interesting aspects of statistical mechanics. The treatise under review is not concerned with these difficulties; although the author, in a clear and illuminating preliminary statement, calls attention to them. The volume is, in fact, not a critical study of the foundations of statistical mechanics, but rather a rich and varied development of the applications of statistical mechanics to problems of modern physics. Some idea of the scope and importance of these applications is gained from the following partial list of subjects treated, taken from the table of contents: assemblies of permanent systems; specific heats of simple gases; partition functions for temperature radiations and for crystals; dissociation and evaporation; the relationship of the equilibrium theory to classical thermodynamics; Nernst's heat theorem and the chemical constants; the theory of imperfect gases; interatomic forces; applications of the equilibrium theory to thermoionics; the dielectric, dia- and paramagnetic constants of gases; the properties of dilute solutions; atmospheric problems; applications to stellar interiors; chemical kinetics in gaseous systems; fluctuations. Any discussion here of the five hundred sixty pages of detailed analysis applied to these problems is both impossible and inappropriate. It must suffice to say that the problems treated are of first rate importance, and that the exposition is clear and accurate, although it naturally demands a considerable amount of physical and mathematical sophistication on the part of the reader. The treatise is a brilliant and monumental piece of work which is certain to rank, for many years, as one of the most useful, complete, and scholarly books ever written in this field.

Although the reviewer does not feel it wise to discuss at this place the more purely physical aspects of this book, there is, nevertheless, one such matter

[1] Infinitesimal, in the accurate sense, as the number of dynamical systems becomes infinite.

of principal importance which should be at least mentioned. It is one of the distinguishing features of this treatise that the quantum aspects of the statistical problems are introduced *ab initio* so that classical statistics appears as a natural limiting case of the more general quantum statistics, rather than that quantum statistics appears as a bizarre departure from classical statistics.

It was mentioned above that Professor Fowler had been interested in this book in applying statistical mechanics, rather than studying statistical mechanics as such. In the process of application, however, he has made a great improvement in technique and has added materially to the elegance, logical accuracy, and ease with which the calculations may be carried out. This matter is of particular mathematical interest and deserves discussion here. To simplify the discussion, suppose that the type of dynamical system under consideration is a gas of given volume, etc., and that one is investigating the law for the distribution of velocities of the various molecules forming one of the systems. Given a distribution of representative points in the phase space, one calculated, according to the older procedure, the most probable distribution of velocities; meaning the distribution of velocities which would most probably be found to obtain in a gas sample picked at random from the ensemble of samples. One then argued that this most probable law for the distribution of velocities was not only the most probable but that it (or one differing negligibly from it) was characteristic of all but an infinitesimal fraction of the systems forming the ensemble. When these two points had both been argued, one called (following Jeans) the property in question a "normal" property.

Of the two steps just noted, the latter offers considerably more difficulty, and it has become not infrequent for authors to omit this portion of the argument entirely, partly because it is hard to make it accurate and partly because of the general optimistic belief, based on experience, that a most probable property is likely to be also a normal property. Professor Fowler, using the new procedure inaugurated by Darwin and himself, considers the normal properties of an ensemble to be the average properties rather than the properties of maximum frequency of occurrence; and in the calculation of these average properties, Darwin and Fowler have introduced a most clever and convenient method of calculation. The calculation of these averages, in fact, requires the calculation of complicated sums of factorials. In the older procedure, such factorials were approximated by the Stirling formula, and the disagreeable task of determining the degree of approximation in the final results was usually neglected. A typical, although simple, example of the sort of factorial sums which enter is

$$C = \sum_{a,b} \frac{M!}{a_0!a_1!\cdots} \frac{N!}{b_0!b_1!\cdots},$$

summed for positive or zero integral values of a and b, subject to

$$\sum_r a_r = M, \quad \sum_s b_s = N, \quad \sum_r r a_r \epsilon + \sum_s s b_s \eta = E,$$

where M, N, ϵ, η, and E are constants whose physical meaning need not concern us here. Darwin and Fowler noticed that C is equal to the coefficient of z^E in

$$(1 - z^{\epsilon})^{-M}(1 - z^{\eta})^{-N},$$

which coefficient, in turn, may be written as

$$\frac{1}{2\pi i}\int \frac{dz}{z^{E+1}} \frac{1}{(1 - z^{\epsilon})^{M}(1 - z^{\eta})^{N}},$$

the path of integration circulating once counter-clockwise around $z=0$. So far the calculation is exact; but in the actual physical cases one is interested in the asymptotic value of this integral as M, N, and E tend toward infinity in fixed ratios. This asymptotic value can be readily computed by the so-called method of steepest descent.[2] This method which first came into prominence when used by Debye in 1909 to obtain asymptotic expressions for certain cylinder functions, depends upon the fact that the integrand above has, along the real axis, a minimum at a point $z=\theta$ between $z=0$ and $z=1$; this same point being, however, the location of a sharp maximum as one traverses the circle, with center at the origin, which passes through $z=\theta$. If one uses this circle as the contour along which the above integral is to be extended, then the value of the integral arises, sensibly entirely, from the immediate neighborhood of $z=\theta$. The asymptotic value of the integral, and hence the original factorial sum, can thus be computed with great simplicity and with complete rigor. This elegant method is used with great skill and effectiveness throughout the volume.

WARREN WEAVER

Statistics. By William Vernon Lovitt and Henry F. Holtzclaw. Prentice-Hall, Inc., New York, 1929. 304 Pages. $4.00

Mathematics Preparatory to Statistics and Finance. By George N. Bauer. The Macmillan Co. New York, 1929. 337 Pages.

The Lovitt and Holtzclaw text is rather unique in that it could be used either in a first course in mathematical statistics or in a course in statistics in business schools. In the latter case, the author's claim that "some of the more difficult mathematical passages may be omitted without destroying the continuity of the course" would probably be accepted fairly often, while in the former case, much of the first six chapters would have to be served in concentrated doses.

The text includes the usual treatment of averages and dispersion and not only an unusually full treatment of the theory of correlation itself but also references to the various modifications of the Pearsonian coefficient.

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[2] Called, in German, "die Sattelpunktmethode"; and, in French, "méthode du col."

Index numbers and the standard treatment of time series are given an unusual amount of attention. The text includes also appropriate parts of college algebra and simple curve fitting including the normal probability curve and its bearing upon the meaning and use of various types of probable errors. The book has been carefully prepared; only a few errors, of a trivial character, were noted.

The Bauer text is essentially a very acceptable elementary text covering both analytic geometry and college algebra but draws fairly consistently upon the general field of finance and statistics for its applications and exercises. While a little of the theory of finance and of statistics is included it is admittedly of a very elementary character.

Some topics of a more advanced character, such as least squares, are taken up in a very mechanical or rule-of-thumb way but even though the treatment is not complete in such cases this fact is frankly admitted and the work is evidently included mainly for the sake of a much desired drill and not as an essential part of the fundamental theory.

Logarithms are treated in great detail and, in general, the whole text is well adapted and seems to have been prepared much for those who are a little "slow" in their mathematics.

While the reviewer is not entirely in sympathy with the use of terms like "the straight line law," "the law of the parabola," "accumulated amount" (for the amount of an annuity), or with the definition of the coefficient of correlation which "is not the usual definition," etc., the purpose back of them is recognized and, for the most part, good and by the time a student finishes this text he will not only be well "prepared" for a regular course in finance or in statistics but also probably very much interested.

C. H. FORSYTH

An Introduction to Mechanics. By J. W. Campbell. Houghton Mifflin Company, Boston, 1929. XIV+384 pages. $3.50.

This is an admirable text on the subject. Its appeal will probably be more direct to the young student of mathematics, although the young engineering student would profit immensely if the opportunity were offered to him to elect it after his first course in the calculus.

The author begins his preface thus: "The statement has been made and does not seem to have been challenged, that the elementary principles of mechanics are more poorly understood by students than the elementary principles of any other subject." Later he observes: "In my opinion a fundamental principle when presented should be in such form that, although a student will probably on a first study of it get only a few of its important features, he will when occasion later arises find that the treatment stands analysis. His appreciation of it will grow and he will have nothing to unlearn." The author does not lose sight of these ideas in the development of his text.

The calculus is freely and naturally used throughout. Topics recur instead of being "exhausted" upon their first presentation. Relative motion, for example, is introduced in the first chapter and further developed as late as chapter XIII. So with other topics. The catenary types receive especial attention and are presented in an unusually attractive manner. The author happily elected to solve the problem of the freely hanging pulley which is itself in motion, being attached to a fixed pulley—a problem illustrating so well the fundamental relation between force and motion. There are many excellent features—large, clear type, well drawn figures, more than six hundred well selected problems and, best of all, the delightful manner of the author's presentation of his subject.

At the end there are eight appendices: principles of computation, definite integrals, "force", Simpson's rule and other topics, closing with interesting remarks on the growth and conception of mechanics.

One cannot predict how a new book will work out with a class until he has used it at least once, but this book bids fair to succeed if one may judge from its content and manner of presentation.

G. H. HUNT

Numerical Tables of Hyperbolic and Other Functions. By J W. Campbell. Houghton Mifflin Company, Boston, 1929. 76 pages. $1.25.

The tables by Professor Campbell are fourteen in number and include hyperbolic functions (four decimal places and five significant figures), $(\sinh x)/x$, $(\cosh x-1)/x$, $(\cosh x)/x$, trigonometric functions for both radian and degree values of the angle, $\log_{10}x$, $\log_e x$, x^2, x^3, $1/x$, together with notes on the tables and a useful tabulation of catenary and transmission line formulae, also formulae for solving cubic equations, with remarks on each. These tables would naturally accompany the author's text on Mechanics, but are valuable in themselves. The type is good.

G. H. HUNT

Elementary Mechanics. By Joseph B. Reynolds. Prentice-Hall, Inc., New York, 1928. VIII+250 pages. $2.50.

In this text the leading topics of the elements of mechanics are developed without the aid of the calculus. The text, therefore, is especially adapted to classes of students who have not studied the calculus. It is, however, not a shop book or in any sense a handbook of rules. The mathematical principles are remarkably well presented. Algebra, trigonometry and geometry are freely used and, to replace the calculus, the principles of limits. The selection of problems and in general the discussion of the subject emphasizes the engineering side of mechanics.

The chapters: resolution and composition of forces, equilibrium of rigid

bodies, center of gravity, friction, work, kinematics, kinetics, energy, impact and simple harmonic motion, indicate the order of treatment.

The author says in his preface: "This text in elementary mechanics is intended particularly for students who desire a working knowledge of the easier applications of mechanics" . . . . . "An attempt has been made to avoid problems involving complicated arithmetic because of the data given." How many students have lost all sense of principle involved when in beginning a new subject problems were too frequently introduced which served merely to bewilder them in a maze of complicated numerical calculations.

This book is superior to most texts which have substituted the language of the theory of limits for the symbols of the calculus. In that the author has evidently conscientiously striven to present his theory rigorously the question arises as to whether it will prove successful with the classes for which it is intended. There is every reason to suppose that this will be the case.

G. H. HUNT

Plane Trigonometry. By J. B. Rosenbach and E. A. Whitman. John Wiley & Sons, New York, 1929. 216 pages. $2.00

The advertising circulars of this book emphasize that it is a thoroughly teachable book. The teachable and teaching qualities are those which have impressed the reader as being the most outstanding of its good features. It is easily seen that the text is the result of long classroom experience. Detailed proofs and explanations are given for theorems and illustrative examples and there are many diagrams with more than the usual number of significant details. The exercises, with answers at the back for the odd numbered ones, are numerous, varied, and well arranged. Thus, during the course of a chapter there are examples for drill and written work, and at the end of the chapter a set of general exercises.

Besides the topics usually found in a text on trigonometry, there are sections on the periodicity of trigonometric functions; various types of trigonometric and inverse trigonometric equations; drawing graphs by the composition of ordinates, and graphic solutions of equations involving trigonometric functions of one angle. There is a chapter on the theory and use of logarithms, but complete logarithmic tables are not included.

In the first chapter the trigonometric ratios are defined for the general angle. Later these definitions are restated, for the acute angle, in terms of the sides of a right triangle. Radian measure follows directly after the definition of the (general) angle. At the end of the book tables are given converting degrees, minutes, and seconds to radians, and inversely,—also a table of sines, cosines, and tangents for angles expressed in radians.

The only objection which has occurred to the reader is that important facts and theorems do not stand out with sufficient emphasis among the great number of illustrative examples and the variety of types of cases given. The

text would not be recommended for use in a general course which contains a unit of trigonometry, but it should prove most satisfactory for colleges and schools at which a full course in trigonometry is offered.

LAURA GUGGENBÜHL

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3409. *Proposed by Norman Anning, University of Michigan.*

When $3b^2>8ac$, the curve, $y=ax^4+bx^3+cx^2+dx+e$, has two real inflectional tangents. Prove that they cut off equal areas.

3410. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Construct a triangle given the base, the difference of the base angles, and the bisector of the vertical angle.

3411. *Proposed by J. Rosenbaum, Milford, Conn.*

A particle within a closed plane curve is attracted by every point of the curve with a force proportional to the distance from the particle to the point. Find the position of equilibrium of the particle.

UNSOLVED PROBLEMS

2954 [1922, 81]. *Proposed by C. N. Mills.*

A machine-gun is placed on an armored train which is moving with a velocity v feet per second along a straight horizontal track. The muzzle velocity of the bullets is v feet per second. Find the greatest range, (1) in front and (2) behind the train.

3034 [1923, 276]. *Proposed by J. L. Riley, Stephenville, Texas.*

If every root of the equation $f'(x)=0$ be subtracted from every root of the equation $f(x)=0$, find the sum of the reciprocals of the differences.

3124[1925, 138]. *Proposed by M. B. Porter, University of Texas.*

$f(x)$ and $\phi(x)$ are polynomials and all the zeros of $\phi(x)$ are real. Let $P(x) = \phi(x)[f^{(i)}(x)]^r$ where $f^{(i)}$ stands for the ith derivative of $f(x)$.

Prove that $[P'(0)]^2 < P''(0)P(0)$ (1) is a sufficient condition that $f(x)$ has imaginary zeros. Newton's test[1] $C_i^2 \leqq C_{i-1}C_{i+1}$ where C_i is the coefficient of x^i in $f(x)$ is a special case of (1).

SOLUTIONS

3045[1923, 449]. *Proposed by S. A. Corey, Des Moines, Iowa.*

If in the equation

$$s!\left[\frac{1}{(s+1)!} + \frac{c_3}{4!(s-1)!} + \frac{c_5}{6!(s-3)!} + \frac{c_7}{8!(s-5)!} + \cdots + \frac{c_s}{(s+1)!2!}\right] = 0 \quad (1)$$

$c_3, c_5, c_7, \cdots c_s$ be given and retain such constant values that (1) is satisfied for all positive odd integral values of s, $(s>1)$, prove that if s be decreased by unity (so that $s=2n$), then the left member will become equal to $\pm B_n$, according as n is odd or even, B_n being Bernoulli's nth number. Also show how any one of the constants c may be found without first finding all the preceding constants.

Solution by Otto Dunkel, Washington University.

The Bernoulli numbers, B_n, occur in the coefficients of a number of expansions of functions, and we shall use the following expansion for the derivation of the required relations

$$(1) \qquad y = \frac{x}{e^x - 1} = 1 - \frac{1}{2}x + B_1\frac{x^2}{2!} + \cdots + (-1)^{i-1}B_i\frac{x^{2i}}{(2i)!} + \cdots .$$

Writing $y(e^x-1)=x$ and differentiating both sides m times, we obtain

$$(2) \qquad e^x[y + {}_mC_1y' + \cdots + {}_mC_{m-1}y^{(m-1)}] + (e^x - 1)y^{(m)} = 0, \quad m \geqq 2,$$
$$= 1, \quad m = 1,$$

where $y^{(i)} = d^iy/dx^i$ and ${}_mC_i$ is the binomial coefficient. Setting $x=0$, $y_0=1$, $y_0' = -\frac{1}{2}$, $y_0^{(2i+1)} = 0$, and $y_0^{(2i)} = (-1)^{i-1}B_i$, we shall make use of three equations obtained by setting $m=2n$, $2n+1$, $2n+2$. The first of these equations is

$$(3) \quad 1 - \tfrac{1}{2}{}_{2n}C_1 + {}_{2n}C_2B_1 + \cdots + (-1)^{i-1}{}_{2n}C_{2i}B_i + \cdots + (-1)^{n-2}{}_{2n}C_{2n-2}B_{n-1} = 0,$$

while the second and third are similar but contain a final term in B_n. Multiply the first equation by $2n+1$, the second by $-2n$, and add the results; divide this result by $(2n+1)!$, and split the last term into two terms by writing the $2n$ in its coefficient as $(2n-1)+1$. There results:

[1] See Netto's *Vorlesungen über Algebra* (1896), vol. 1, p. 234, for an account of Sylvester's paper.

$$(4)\quad \frac{1}{(2n+1)!}-\frac{B_1}{(2n-1)!2!}+\frac{3B_2}{(2n-3)!4!}-\cdots + (-1)^n\frac{(2n-1)B_n}{1!(2n)!}+(-1)^n\frac{B_n}{(2n)!}=0.$$

Now multiply the second equation by $2n+2$, the third by $-(2n+1)$, add the results, and divide the resulting equation by $(2n+2)!$. We shall thus obtain a second equation which may be written down from the above by omitting the last term and by increasing by unity the numbers in the first factorial factors of the denominators. By comparing these results with the equations of the problem we see that $c_{2i+1}=(-1)^i(2i-1)\ (2i+1)\ (2i+2)B_i$, and it follows that any method of calculating the B's gives by a very slight modification a method of calculating the c's, and conversely. One method is obviously given by solving a set of the linear equations in the B's or the c's by the use of determinants. Or we may proceed thus. Write (1) in the form $y=\log_e[1+(e^x-1)]/(e^x-1)$, and then develop in powers of (e^x-1). We have then

$$(5)\quad y=\sum_{m=0}^{\infty}(-1)^m\frac{(e^x-1)^m}{m+1}.$$

We may now write the expansion of (e^x-1) in powers of x, obtain the mth power of the series, and collect the terms. The coefficient of x^{2n} will then be $(-1)^{n-1}B_n/(2n)!$. Or we may obtain a result which is easier to write by first expanding by the binomial theorem $(e^x-1)^m$ and then developing into series each term of the result, noting that the last term to be so treated is given by $m=2n$. The result may be written

$$B_n=(-1)^n\left[\frac{\Delta_{2n}}{2}-\frac{\Delta_{2n}^2}{3}+\cdots-\frac{\Delta_{2n}^{2n}}{2n+1}\right],$$

where

$$\Delta_{2n}^m=\sum_{\gamma=1}^{m}(-1)^{m-\gamma}{}_mC_\gamma\gamma^{2n}.$$

It is clear from the method that $\Delta_{2n}=1$, $\Delta_{2n}^{2n}=(2n)!$. If we form the first difference $(x+1)^{2n}-x^{2n}$ and then set $x=0$, we shall obtain Δ_{2n}; then forming the second difference $(x+2)^{2n}-2(x+1)^{2n}+x^{2n}$ and setting $x=0$, we shall obtain Δ_{2n}^2; and finally Δ_{2n}^k is the value of the difference of the kth order of x^{2n} after setting $x=0$. We may obtain a homogeneous equation in the B's by multiplying the second equation corresponding to (4) by $-(2n+2)$ and adding the result to the equation written. We then obtain

$$B_1-{}_{2n}C_2B_2+{}_{2n}C_4B_3-\cdots+(-1)^n{}_{2n}C_{2n-4}B_{n-1}+(-1)^{n+1}(2n+1)(n-1)B_n=0.$$

3362 [1929, 105]. *Proposed by R. E. Gaines, University of Richmond.*

A triangle ABC circumscribes an ellipse of axes $2a$ and $2b$ so that A lies on one directrix and B on the other, while AB, BC, CA touch the ellipse at P, Q, R. Show that the envelope of QR and the locus of C are ellipses with $2a$ as the major axis of one and the minor axis of the other, while $2b$ is the mean proportional between the other axes.

Solution by William Hoover, Columbus, Ohio.

Let the equation of the given ellipse be $b^2x^2+a^2y^2-a^2b^2=0$, and the equation of the line AB be $lx+my-n=0$; then for tangency of AB we must have $n^2=a^2l^2+b^2m^2$. A line through the intersection of AB and the directrix $x-ae^{-1}=0$ has the equation $lx+my-n-k(x-ae^{-1})=0$. Applying to this line the condition for tangency above, we find that $ka(1-e^2)=2e(n-ael)$; and inserting this value of k in the equation above, we obtain for AC the equation,

$$[al(1+e^2)-2ne]x+ma(1-e^2)y+an(1+e^2)-2a^2el=0.$$

The equation for BC is obtained by replacing in this equation e by $-e$. Solving the two equations thus obtained, we find

$$l/n=-x/a^2, \quad m/n=-(1-e^2)y/(1+e^2)b^2;$$

and, inserting the values of these two ratios in the equation of condition above, we obtain for the locus of C,

$$x^2/a^2+[(1-e^2)/(1+e^2)]^2(y^2/b^2)=1. \tag{1}$$

This is the equation of an ellipse having for its minor axis the major axis $2a$ of the given ellipse.

Since the envelope of QR is the polar reciprocal of this ellipse (1) with respect to the given ellipse, it must also be a conic, which is evidently an ellipse within the given ellipse and tangent to it at the extremities of its major axis. Denote an extremity of the major axis of (1) by $(0, y_1)$; then the polar of this point with respect to the given ellipse is $yy_1-b^2=0$. Then if $(0, \pm y_2)$ are the extremities of the minor axis of the envelope, we have $y_2y_1=b^2$; and this completes the proof.

This problem is a special case of more general theorems which have interested British geometers such as Salmon, Cayley, and Casey. See Salmon's *Conic Sections*, 6th ed. (1879), pp. 250, 257, 319, 349, 350. In the last place referred to the theory of invariants is used, and there is reference to the Quarterly Journal of Mathematics, vol. 1 (1857).

Also solved by Lawrence Hampton and the proposer.

3367 [1929, 169]. *Proposed by Harry Langman, Arverne, L. I., N. Y.*

Given any triangle. On each side construct an equilateral triangle externally. The centers of these triangles determine another equilateral triangle A. Similarly an equilateral triangle B is determined by constructing the equilateral

triangle internally. Show that the difference between the areas of the triangles A and B is equal to the area of the given triangle.

Solution by J. W. Peters, University of Illinois.

Let x_1, x_2, x_3, representing any three points in the complex plane, be the coordinates of the vertices of the given triangle. Let y_1 be the point forming with x_2 and x_3 the equilateral triangle drawn externally on the side x_2x_3. Since the triangle y_1, x_2, x_3 is equilateral,

$$\begin{vmatrix} y_1 & x_2 & x_3 \\ 1 & \omega^2 & \omega \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

where ω and ω^2 are the complex cube roots of unity. From this relation, we find $y_1 = -\omega(x_3+\omega x_2)$. Similarly, if y_2 and y_3 are the vertices of the equilateral triangles drawn on the sides x_3x_1 and x_1x_2, respectively, we have:

$$y_2 = -\omega(x_1+\omega x_3) \text{ and } y_3 = -\omega(x_2+\omega x_1).$$

If a_1, a_2, a_3 are the centroids of the three equilateral triangles, then

$$a_1 = \tfrac{1}{3}(1-\omega)(x_3 - \omega^2 x_2),$$

$$a_2 = \tfrac{1}{3}(1-\omega)(x_1 - \omega^2 x_3),$$

$$a_3 = \tfrac{1}{3}(1-\omega)(x_2 - \omega^2 x_1).$$

These are the vertices of triangle A. By finding the quotients like

$$(a_1-a_2)/(a_3-a_2)$$

we see that the angles of A are all equal. Hence, A is equilateral. The area is given by

$$-4iA = \left| a\ \bar{a}\ 1 \right| = \begin{vmatrix} a_1 & \bar{a}_1 & 1 \\ a_2 & \bar{a}_2 & 1 \\ a_3 & \bar{a}_3 & 1 \end{vmatrix},$$

where the bars denote the conjugate quantities.

If z_1, z_2, z_3 are the vertices of the equilateral triangles drawn internally, we have $z_1 = -\omega(x_2+\omega x_3)$, $z_2 = -\omega(x_3+\omega x_1)$, and $z_3 = -\omega(x_1+\omega x_2)$. If b_1, b_2, b_3 are the centroids of these three equilateral triangles, then

$$b_1 = \tfrac{1}{3}(1-\omega)(x_2 - \omega^2 x_3),$$

$$b_2 = \tfrac{1}{3}(1-\omega)(x_3 - \omega^2 x_1),$$

$$b_3 = \tfrac{1}{3}(1-\omega)(x_1 - \omega^2 x_2),$$

These are the vertices of triangle B. It is easily verified that B is equilateral. The area of B is given by $4iB = \left| b\bar{b}1 \right|$.

If in the equations giving the areas of the triangles A and B, we replace

the a's and b's by their values in terms of the x's, and expand the determinants, we find on subtracting that

$$-4i(A-B) = \left| x\ \bar{x}\ 1 \right| .$$

The determinant on the right is $-4iC$, where C is the area of the given triangle.

Dr. H. W. Bailey, my colleague, suspected some further interesting facts about this configuration, which we have verified. The triangles A and B have the same centroid and it is the centroid of the given triangle. Furthermore the angle of intersection of the triangles A and B is invariant under homologies, i.e., transformations of the type $y=ax+b$. The proof follows: The angle of intersection is given by the amplitude of $(b_2-b_3)/(a_2-a_3)$ or $-(x_1+\omega x_2+\omega^2 x_3)/(x_1+\omega^2 x_2+\omega x_3)$. The quantities $x_1+\omega x_2+\omega^2 x_3$ and $x_1+\omega^2 x_2+\omega x_3$ are the Lagrange resolvents of the cubic equation which has x_1, x_2, x_3 for its roots. The quotient of these two resolvents is invariant under homologies. Hence the angle is invariant.

Also solved by W. E. Buker, J. W. Clawson, P. J. Federico, W. W. Johnson, J. H. Neelley, and A. Pelletier.

3368 [1929, 169]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

About a given quadrilateral to circumscribe a rhombus similar to a given rhombus.

Solution by L. W. Johnson, The University of Oklahoma.

Let $ABCD$ be the given quadrilateral and $KLMN$ the given rhombus. On AB and CD construct segments of circles external to $ABCD$ in which the equal angles K and M of the given rhombus may be inscribed. Let E and F, respectively, be the mid-points of these arcs and E' and F', respectively, the mid-points of the arcs which complete the circles of which the arcs already constructed are a part. Let the join of E' and F' intersect the arcs upon which E and F lie in the points G' and H', respectively. Join G' to A and B, and H' to C and D, calling R the intersection of $G'A$ with $H'D$ and S the intersection of $G'B$ with $H'C$. $G'H'RS$ is the required rhombus.

Proof: $G'H'RS$ is a parallelogram since its opposite sides make equal angles with $G'H'$. It is a rhombus since the triangle $G'H'S$ is isosceles.

Discussion: The join of E and F determines two points G and H analogous to the points G' and H' used above which may therefore be joined, respectively, to A and B and to C and D to determine another rhombus satisfying the conditions of the problem. Since what has been said of one pair of opposite sides of the given quadrilateral applies to the other pair, there are in general four solutions to this problem. If G' and H' coincide the number of solutions will be infinite.

Also solved by A. Pelletier.

3370 [1929, 169]. *Proposed by Paul Wernicke, Washington, D. C.*

Write down an orthogonal transformation from rectangular Cartesian coordinates X, Y, Z to x, y, z having the same origin such that the z-axis be-

comes the line $X=Y=Z$ and that the y-axis lies in the plane through the Y and z axes.

Solution by J. H. Neelley, Carnegie Institute of Technology.

Let $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3$ be, respectively, the direction angles of the three mutually perpendicular lines ox, oy, oz. Then the six conditions on these nine angles together with the special conditions given in the problem make $\alpha_3=\beta_3=\gamma_3$ and $\alpha_2=\gamma_2$ with $\beta_2=\frac{1}{2}\pi\pm\beta_3$ and $\alpha_1=\pi-\gamma_1$, $\beta_1=\frac{1}{2}\pi$. So the possible transformations are

$$\begin{aligned} X &= ax + by + cz, \\ Y &= -2by + cz, \\ Z &= -ax + by + cz, \end{aligned}$$

where $a=\pm 2^{-1/2}$, $b=\pm 6^{-1/2}$ and $c=\pm 3^{-1/2}$. All groupings of the signs are possible.

Also solved by A. Pelletier.

3371 [1929, 169]. *Proposed by Harry Langman, New York City.*

Let ABCD be any simple quadrilateral (convex or cross) inscribed in the circle whose center is O. Let AB and DC meet in F, BC and AD in E. Let M be the midpoint of the third diagonal, EF, and MU and MV tangents at U and V. Let EU and FV meet in P; EV and FU meet in Q. Take the point G on EF so that $\angle DGF=\angle DAF$, and let AC cut OG in the point R. Let the secants GA, GB, GC, GD, cut the circle again in the points A', B', C', D', respectively. Take $OA=r$. Then prove the following:

(a) G is the Clifford point of the quadrilateral $ABCD$—i.e., the common intersection of the circles about the four possible triangles formed by the sides of the quadrilateral.

(b) The square of EF is equal to the sum of the squares of the tangents to O from E and F (Casey: *Sequel to Euclid*).

(c) The circle on EF as diameter cuts the circle O orthogonally (Casey).

(d) $AC'//BD'//A'C//B'D//EF\perp OG$, proving the theorem that the Clifford point of an inscribed quadrilateral is the foot of the perpendicular from the center to the third diagonal.

(e) P and Q are the intersections of OG and the circle O.

(f) BD and UV pass through R, proving the theorem that the perpendicular from the center on the third diagonal of an inscribed quadrilateral passes through the intersection of the other two diagonals.

(g) If a quadrilateral be circumscribed about a circle at the vertices of an inscribed quadrilateral, the two pairs of diagonals intersect in a common point.

(h) Any obtuse-angled triangle may be the self-conjugate triangle of an inscribed quadrilateral. If the triangle be given, the center and radius of the circle are determined; but, when one side of the triangle has been chosen as the third diagonal of an inscribed quadrilateral, the quadrilateral is not thereby determined, there being one degree of freedom.

Solution by J. W. Clawson, Ursinus College.

(a) Assume $\angle DCB \geqq \angle DAB$. $\angle EGD = \angle BAD = \angle ECD$. Hence D, E, G, C are concyclic. Similarly it may be shown that the other groups of four points are concyclic. Hence G is the "Clifford" or "Miquel" or "focal" point of the quadrilateral $ABCD$.

(b) This theorem applies only to the case of a convex quadrilateral. $ED \cdot EA = EG \cdot EF$, since A, D, G, F are concyclic. $FB \cdot FA = FG \cdot FE$, since A, B, G, E are concyclic. Hence, the square of the tangent from E to the given circle plus the square of the tangent from F to the given circle $= EG \cdot EF + FG \cdot FE = EF^2$.

(c) Draw OE and let K be the foot of the perpendicular from F to OE. Then FK is the polar of E, since E and F are conjugate points with respect to (O). Hence $OE \cdot OK = r^2$. Let the circle (M) on EF as a diameter cut (O) in U and V. Since it passes through K, $OK \cdot OE$ is also the square of the tangent from O to (M). Hence $OU = OV = r$ are tangents to (M). This shows that any circle having two conjugate points of the circle (O) as extremities of a diameter is orthogonal to (O).

(d) From the inscribed quadrilateral $DCBD'$ we have $\angle BD'D = \angle BCD (= \angle DCE)$; and since E, G, C, D are concyclic $\angle DCE = \angle DGE$. Hence $BD' \| EF$. Similar proofs apply to AC', DB', CA'. Since BD' and DB' are parallel chords of (O) and DD' and BB' meet in G, the triangles GBD' and $GB'D$ are isosceles. Hence the perpendicular from G to these chords bisects them and passes through O. Therefore GO is perpendicular to EF.

(e) A circle can be passed through U, V, P, Q; and from the complete quadrilateral we see that E and F are conjugate with respect to it. Therefore the circle (M) is cut orthogonally by it at U and V; this circle must therefore be (O), and PQ is a diameter. In the triangle EFP, the altitudes FU and EV meet in its orthocenter Q; hence POQ is the third altitude perpendicular to EF. From (d) we see that it cuts EF in G.

(f) The pole of EF is determined by the original quadrilateral as the intersection R of BD and AC. From the quadrilateral $UVPQ$, it follows that the pole R must also be the intersection of PQ and UV.

(g) If tangents at A and B intersect in I, at B and C in J, at C and D in K, at D and A in L, the polars of K and I pass through F, hence the polar of F is KI. But, since the polar of R passes through F, the polar of F passes through R. Hence KI passes through R. Similarly LJ passes through R.

(h) EFR is a self-conjugate triangle. O is the orthocenter of the triangle. $OR \cdot OG = r^2$.

Hence, starting with any obtuse-angled triangle EFR, locate the orthocenter O. Let OR cut EF at G. Find the mean proportional between OR and OG. This gives r, the radius of the circle. Now draw any line through E cutting the circle at D and A. Join DF, AF cutting circle at C and B. Let ER cut DC, AB in H and N. Then $(ANBF)$ and $(DHCF)$ are harmonic ranges. Therefore AD, NH, BC are concurrent. Hence BC passes through E.

Also solved by A. Pelletier.

3372 [1929, 232]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Given a point of a conic, the tangent at this point, an axis, and the center (at a finite, or infinite distance), the following two lines are constructed: (1) the symmetric of the given tangent with respect to the parallel to the given axis through the given point, and (2) the diameter through the symmetric of the given point with respect to the axis. Prove that if at the point of intersection of these two lines the perpendicular is erected to the first line, it will meet the normal to the conic at the given point in the center of curvature of the conic at this point.

Solution by Margaret M. Young, Hunter College of the City of New York.

Consider the ellipse $b^2x^2+a^2y^2=a^2b^2$ and one of its points (x_0, y_0); then the center of curvature for this point has the coördinates $X=x_0-m(1+m^2)/m'$, $Y=y_0+(1+m^2)/m'$, where $m=y_0'$, $m'=y_0''$. Here $b^2x_0+a^2y_0m=0$, and by the elimination of b we find $m(x_0^2-a^2)=x_0y_0$. Solving this last equation for a^2 and differentiating the result we find $x_0y_0m'=m(y_0-mx_0)$. Inserting this value of m' we obtain

$$(mx_0 - y_0)X = mx_0(x_0 + my_0), \qquad m(mx_0 - y_0)Y = -y_0(x_0 + my_0).$$

The equation of line (1) is $y-y_0=-m(x-x_0)$; while that of line (2) is $x_0y=-xy_0$. Their point of intersection (x_1, y_1) is given by $x_1(mx_0-y_0)=x_0(y_0+mx_0)$, $y_1(mx_0-y_0)=-y_0(y_0+mx_0)$. The equation of the line perpendicular to (1) at (x_1, y_1) is

$$m(mx_0 - y_0)y + my_0(y_0 + mx_0) = (mx_0 - y_0)x - x_0(y_0 + mx_0).$$

The values of X and Y given above satisfy this equation and the proof of the construction is complete. The proof is similar for the other two forms of the conic.

Also solved by A. Pelletier.

Note by the Editors. This construction easily follows from the theorem: *The angle between the common tangent and the common chord of a conic and its osculating circle is bisected by the perpendicular from the point of contact to an axis of the conic.* A proof is given in a Note [1924, 51] following the solution of 2990 [1922, 356]. If P is the given point on the conic and PT is the tangent at this point cutting in T the axis AO of the conic, and if P' is the point of the conic symmetric to AO with respect to this axis, then $P'T$ is tangent at P', and the diameter $P'O$ bisects all chords parallel to $P'T$. Hence if we draw the chord PC of the conic parallel to $P'T$, this chord will also be the chord of the osculating circle by the above theorem, and it will be cut in its middle point M by $P'O$. Therefore the perpendicular to PC at M cuts the normal at P in the center of curvature I of the conic.

THE INFORMATION BUREAU FOR APPOINTMENTS

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NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

At the thirty-sixth annual meeting of the American Mathematical Society, which opened at Lehigh University on December 26, Professor W. C. Graustein, of Harvard University, and Professor E. P. Lane, of the University of Chicago, were elected Vice-presidents.

Princeton University announces the appointment of Dr. J. Von Neumann, of the University of Berlin, and Dr. E. Wigner, of the Kaiser Wilhelm Institut für Physikalische Chemie, as visiting lecturers in Mathematical Physics for the second semester of the present academic year.

The one thousand dollar prize offered annually by the American Association for the Advancement of Science for a noteworthy contribution to science presented at a meeting was awarded at Des Moines to Professor Arthur J. Dempster, of the University of Chicago, for work on wave characteristics of the proton.

Professor Arthur Haas, of the University of Vienna, and F. G. Donnan, of the University of London have been invited by the Gibbs Committee (representing the Departments of Physics and Chemistry of Yale University) to be editors of a commentary on the works of J. Willard Gibbs.

On account of the large sale of Cajori's "History of Mathematical Notations," both in Europe and America, The Open Court Publishing Co. expects to publish a new edition next summer.

At Syracuse University Professor Alan D. Campbell has been made head of the department of mathematics in the College of Liberal Arts.

Professor E. F. Cox, of West Virginia Collegiate Institute, has been appointed to an associate professorship at Howard University.

Assistant Professor E. B. Lytle has been promoted to an Associate Professorship of the teaching of mathematics at the University of Illinois.

Dr. G. M. Merriman has been appointed assistant professor of mathematics at the University of Cincinnati.

Mr. C. A. Spicer has been appointed professor of mathematics at Western Maryland College.

Professor J. C. Tinner, of Wilberforce University, has been appointed to a Professorship at Bishop College.

The following appointments to instructorships are announced:
Cornell University, Mr. J. M. Clarkson
Crane Junior College, Miss Edna M. Feltges
Houston Junior College, Mr. W. A. Rees
Union College, Mr. A. H. Fox and Mr. F. W. Lerch
State College of Washington, Mr. John Biggerstaff and Mr. L. G. Butler
Yale University, Dr. A. K. Mitchell
Hunter College of the City of New York, Mr. A. D. Bradley and Miss Harriet Griffin
South Dakota State College, Mr. H. B. MacDougal, Mr. Gordon Fuller, and Miss Ruth Rasmusen

Dr. Edward Drake Roe, Jr., head of the mathematics department of the College of Liberal Arts of Syracuse University, died suddenly from a heart attack, on December 11, at the age of 70. Dr. Roe was graduated from Syracuse University, received an A.M. degree at Harvard and a Ph.D. degree at Erlangen (doing much of his work there under Gordan). He taught at Harvard, Boston University, and Oberlin College before coming to Syracuse. He was a professor at Syracuse from 1900 until his death; since 1919 he was also director of the astronomical observatory and head of the mathematics department. He founded the mathematical fraternity of Pi Mu Epsilon in 1914. He published many papers on mathematical research (principally in algebra), several papers on astronomy and on philosophy, also a text-book in trigonometry and one in algebra. He always stood for high scholarship. He was a very thorough teacher, leaving a lasting impression on the students who took his courses and inspiring in them his own deep love for mathematics and for astronomy.

The Rhind Mathematical Papyrus is Published

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Individual and institutional members may procure copies at $15.00 per set through Secretary Cairns at Oberlin, Ohio. All others must order through the Open Court Publishing Company, 339 E. Chicago Avenue, Chicago, Ill., at $20.00 per set.

IT IS A MAGNIFICENT WORK and should be in every college library. The edition is absolutely limited. Nearly one-third of the sets are already sold, and no more will be available when this edition is exhausted.

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The Chauvenet Prize

In the year 1925, the ASSOCIATION established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period 1929-1931.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

CONTENTS

The Mathematical Association of America. A List of New Members 43

The Thirteenth Annual Meeting of the Missouri Section. By PAUL R. RIDER 43

The Fourth Annual Meeting of the Philadelphia Section. By P. A. CARIS. 45

The Undergraduate Mathematical Curriculum of the Liberal Arts College. By F. L. GRIFFIN 46

The Problem of Two Bodies and a Related Sextic. By W. H. GARRETT 54

The Rectangular Hexagon. By PAUL WERNICKE 59

A Problem in the Calculus of Variations. By C. E. RHODES 63

Some Curious Geometric Properties Connected with Polygenic Functions. By FREDERIC H. MILLER 67

Doubly Homogeneous Functional Equations. By JOSEPH D. GRANT 70

Generalizations of Pascal's and Brianchon's Theorems. By GLENN JAMES 78

QUESTIONS and DISCUSSIONS: Discussion "A note on Horner's method of solving an algebraic equation," by C. W. BRUCE 80

RECENT PUBLICATIONS: Reviews by JAMES BYRNIE SHAW, T. H. HILDEBRANDT, R. A. JOHNSON, F. L. WREN, WARREN WEAVER, C. H. FORSYTH, G. H. HUNT, LAURA GUGGENBÜHL 81

PROBLEMS AND SOLUTIONS: Problems for Solution—3409–3411. Unsolved Problems—2954, 3034, 3124. Solutions—3045, 3362, 3367, 3368, 3370, 3371, 3372 94

The Information Bureau for Appointments 103

NOTES AND NEWS 103

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.

INDIANA, Earlham College, May 2-3.

IOWA.

KANSAS.

KENTUCKY.

LOUISIANA-MISSISSIPPI, Cleveland, Miss., April.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.

MICHIGAN.

MINNESOTA.

MISSOURI.

NEBRASKA, Peru, Neb., May 9.

OHIO, Columbus, Ohio, April 3.

PHILADELPHIA.

ROCKY MOUNTAIN.

SOUTHEASTERN.

SOUTHERN CALIFORIA, Los Angeles, Calif., March 8.

TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE FOURTEENTH ANNUAL MEETING OF THE ASSOCIATION

The fourteenth annual meeting of the Mathematical Association of America was held at Des Moines, Iowa, on Tuesday and Wednesday, December 31, 1929, and January 1, 1930, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. One hundred forty were in attendance at the meetings, including the following one hundred four members of the Association:

HARRIET ANDERSON, Grand Island College
R. C. ARCHIBALD, Brown University
C. H. ASHTON, University of Kansas
C. S. ATCHISON, Washington and Jefferson College

R. W. BABCOCK, DePauw University
FRANCES E. BAKER, Creston (Iowa) Junior College
R. P. BAKER, University of Iowa
A. K. BETTINGER, Creighton University
A. H. BLUE, Western Union College
F. A. BRANDNER, Louisiana State College
W. C. BRENKE, University of Nebraska
M. SUE BURNEY, St. Joseph (Mo.) Junior College
W. H. BUSSEY, University of Minnesota

W. D. CAIRNS, Oberlin College
E. H. CARUS, La Salle, Illinois
E. W. CHITTENDEN, University of Iowa
L. M. COFFIN, Coe College
JULIA T. COLPITTS, Iowa State College
C. E. COMSTOCK, Bradley Polytechnic Institute
I. S. CONDIT, Iowa State Teachers College
BYRON COSBY, Missouri State Teachers College
D. R. CURTISS, Northwestern University

MARIAN E. DANIELLS, Iowa State Teachers College
R. D. DAUGHERTY, University of Iowa
L. L. DINES, University of Saskatchewan

C. W. EMMONS, Michigan State College
H. P. EVANS, University of Wisconsin
H. S. EVERETT, University of Chicago

B. F. FINKEL, Drury College
IRVING FISHER, Yale University
ANNIE W. FLEMING, Iowa State College
M. M. FLOOD, University of Nebraska

M. G. GABA, University of Nebraska
J. S. GOLD, Bucknell University
G. W. GORRELL, University of Denver
C. GOUWENS, Iowa State College